Sequences & Summations

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Sequences & Summations

Though you should be (at least intuitively) familiar with sequences and summations, we give a quick review.

Sequences

Definition

A sequence is a function from a subset of integers to a set S. We use the notation(s):

 $\{a_n\} \quad \{a_n\}_n^\infty \quad \{a_n\}_{n=0}^\infty \quad \{a_n\}_{n=0}^\infty$

Each a_n is called the *n*-th *term* of the sequence.

We rely on context to distinguish between a sequence and a set; though there is a connection.

Notes

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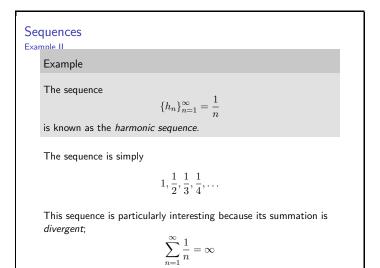
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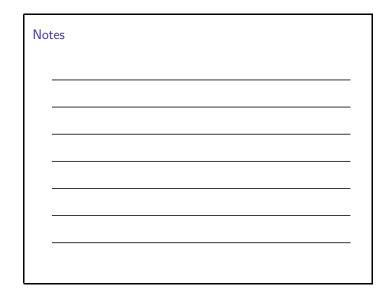
Example	
Consider the see	quence
	$\left\{ \left(1+\frac{1}{n}\right)^n \right\}_{n=1}^{\infty}$
The terms are	
	$a_1 = (1+1)^1 = 2.00000$ $a_2 = (1+\frac{1}{2})^2 = 2.25000$ $a_3 = (1+\frac{1}{3})^3 = 2.37037$ $a_4 = (1+\frac{1}{4})^4 = 2.44140$ $a_5 = (1+\frac{1}{5})^5 = 2.48832$
What is this sec	juence?

Example

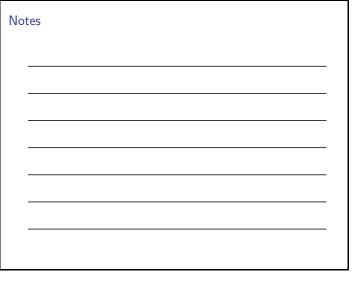
The sequence corresponds to e:

$$\lim_{n \to \infty} \left\{ \left(1 + \frac{1}{n} \right)^n \right\} = e = 2.71828 \dots$$









Progressions I

Definition

A geometric progression is a sequence of the form

 $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$

Where $a \in \mathbb{R}$ is called the *initial term* and $r \in \mathbb{R}$ is the *common ratio*.

A geometric progression is a $\ensuremath{\textit{discrete}}$ analogue of the exponential function

 $f(x) = ar^x$

Progressions II

Definition

An arithmetic progression is a sequence of the form

 $a, a+d, a+2d, a+3d, \ldots, a+nd, \ldots$

Where $a \in \mathbb{R}$ is called the *initial term* and $r \in \mathbb{R}$ is the *common difference.*

Again, an arithmetic progression is a discrete analogue of the linear function, $% \left({{{\bf{n}}_{\rm{s}}}} \right)$

f(x) = dx + a

Progressions III

Example

A common geometric progression in computer science is

$$\{a_n\} = \frac{1}{2n}$$

Here, a = 1 and $r = \frac{1}{2}$

Table 1 on Page 228 (Rosen) has useful sequences.

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Summations I

You should be very familiar with Summation notation:

n

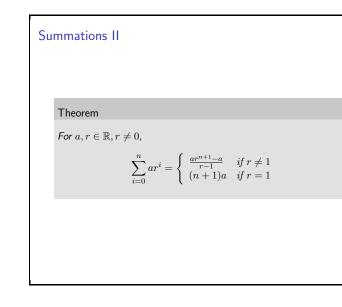
$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \dots + a_{n-1} + a_n$$

Here, j is the index of $\mathit{summation}, m$ is the lower $\mathit{limit},$ and n is the upper $\mathit{limit}.$

Often times, it is useful to change the lower/upper limits; which can be done in a straightforward manner (though we must be careful). $n = n^{n-1}$

$$\sum_{j=1}^{n} a_j = \sum_{j=0}^{n-1} a_{j+1}$$

Sometimes we can express a summation in *closed form*. Geometric series, for example:



Summations III

 $\sum_{i=1}^{n}$

Double summations often arise when analyzing an algorithm.

$$\sum_{j=1}^{i} a_{j} = a_{1} + a_{1} + a_{2} + a_{1} + a_{2} + a_{3} + \dots$$

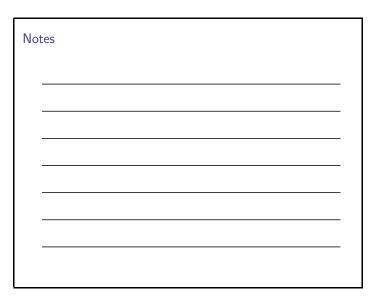
 $a_1 + a_2 + a_3 + \dots + a_n$

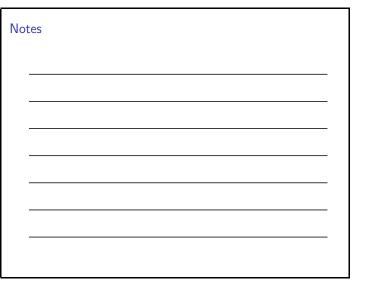
Summations can also be indexed over *elements in a set*.

$$\sum_{s \in S} f(s)$$

Table 2 on Page 232 (Rosen) has useful summations.

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Series

When we take the sum of a sequence, we get a *series*. We've already seen a closed form for geometric series.

Some other useful closed forms include the following.

$$\begin{split} \sum_{i=l}^{u} 1 &= u-l+1, \text{ for } l \leq u \\ \sum_{i=0}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=0}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=0}^{n} i^k &\approx \frac{1}{k+1} n^{k+1} \end{split}$$

Infinite Series I

Though we will mostly deal with finite series (i.e. an upper limit of n for a fixed integer), *infinite series* are also useful.

Example

Infinite Series II

Consider the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

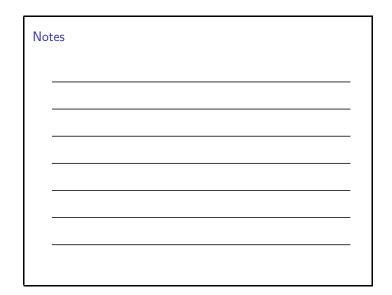
This series converges to 2. However, the geometric series

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots$$

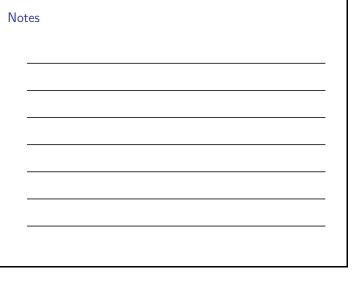
does not converge. However, note that $\sum_{n=0}^n 2^n = 2^{n+1}-1$

In fact, we can generalize this as follows.

Lemma







Infinite Series III

A geometric series converges if and only if the absolute value of the common ratio is less than 1.

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