

## Nebraska Lincoln

#### Universe of Discourse

Predicate Logic and Quantifiers

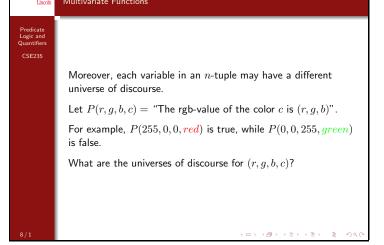
Consider the previous example. Does it make sense to assign to  $\boldsymbol{x}$  the value "blue"?

Intuitively, the *universe of discourse* is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function P(x)= "The test will be on x the 23rd" be?

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#### Quantifiers

Introduction

Predicate Logic and Quantifiers CSE235

A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to *quantify* it. That is, the predicate is true (or false) for *all* possible values in the universe of discourse or for *some* value(s) in the universe of discourse.

Such *quantification* can be done with two *quantifiers*: the *universal* quantifier and the *existential* quantifier.

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### Universal Quantifier

Universe of Discourse

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#### Definition

The universal quantification of a predicate P(x) is the proposition "P(x) is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

If the universe of discourse is finite, say  $\{n_1,n_2,\ldots,n_k\}$ , then the universal quantifier is simply the conjunction of all elements:

$$\forall x P(x) \iff P(n_1) \land P(n_2) \land \cdots \land P(n_k)$$

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## Universal Quantifier Example I

Predicate Logic and Quantifiers CSE235

- ullet Let P(x) be the predicate "x must take a discrete mathematics course" and let Q(x) be the predicate "x is a computer science student".
- $\bullet$  The universe of discourse for both P(x) and Q(x) is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".
- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".



## Universal Quantifier Example I

Logic and Quantifier CSE235

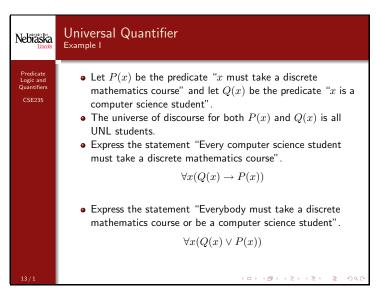
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- $\bullet$  The universe of discourse for both P(x) and Q(x) is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".

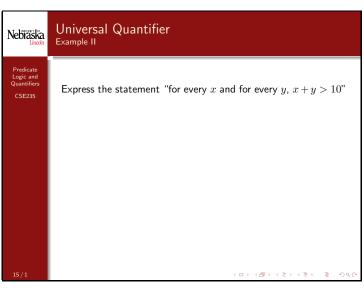
$$\forall x (Q(x) \to P(x))$$

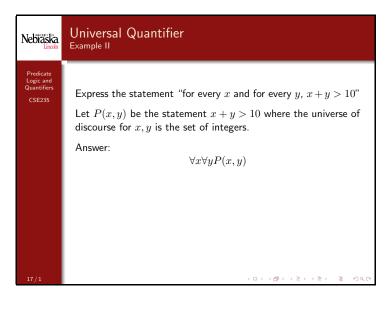
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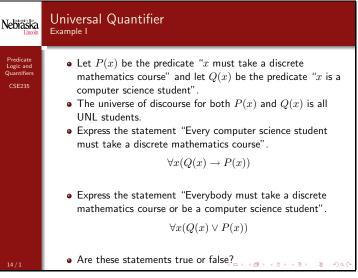
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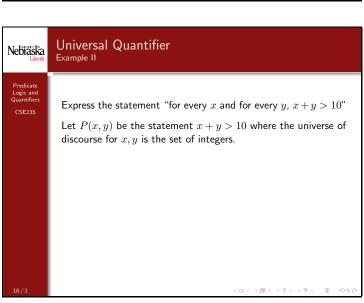
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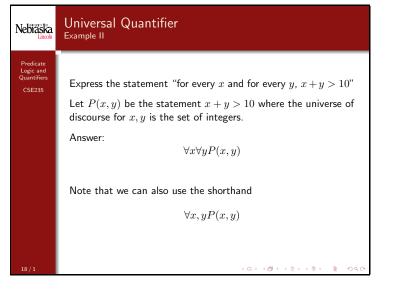


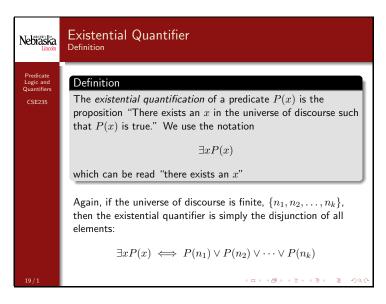


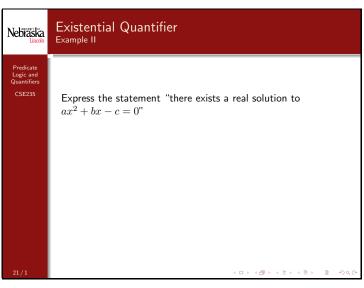


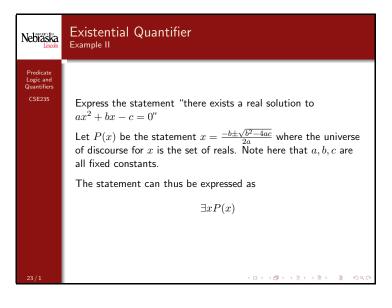


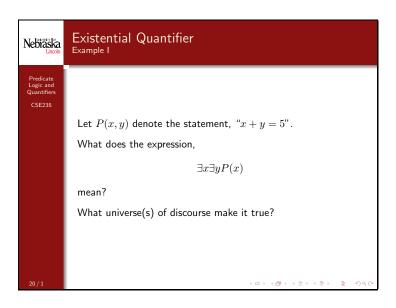


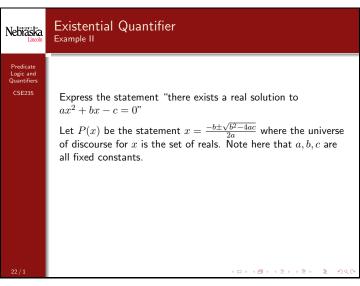


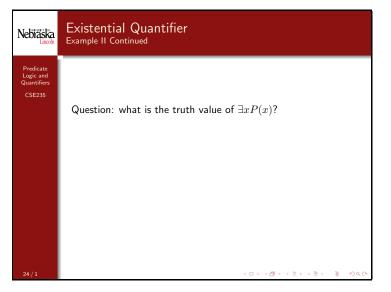


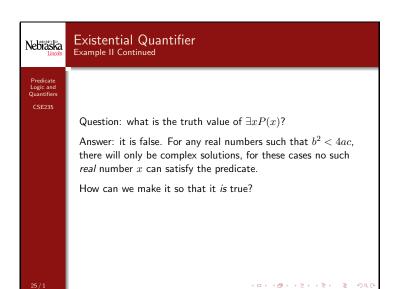












## Question: what is the truth value of $\exists x P(x)$ ? Answer: it is false. For any real numbers such that $b^2 < 4ac$ , there will only be complex solutions, for these cases no such real number x can satisfy the predicate. How can we make it so that it is true? Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$ .

Existential Quantifier

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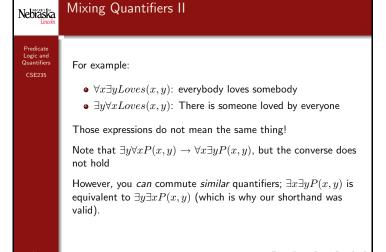
#### Quantifiers Truth Values

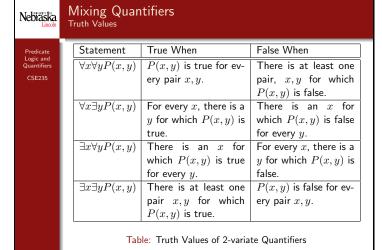
In general, when are quantified statements true/false?

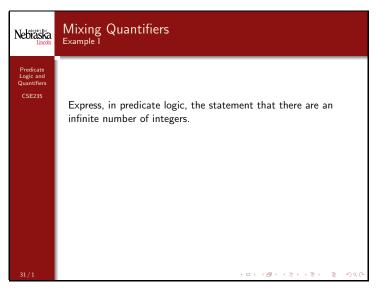
Statement	True When	False When
$\forall x P(x)$	P(x) is true for every	There is an $x$ for
	x.	which $P(x)$ is false.
$\exists x P(x)$	There is an $x$ for	P(x) is false for every
	which $P(x)$ is true.	x.

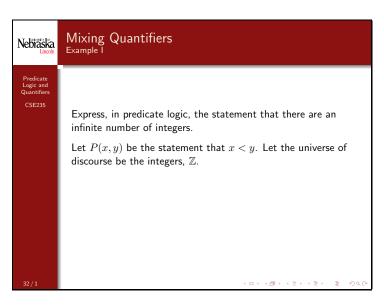
Table: Truth Values of Quantifiers

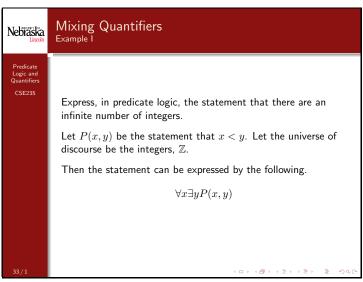
Nebřaska Lincoln	Mixing Quantifiers I
Predicate Logic and Quantifiers CSE235	Existential and universal quantifiers can be used together to quantify a predicate statement; for example,
	$\forall x \exists y P(x,y)$ is perfectly valid. However, you must be careful—it must be read left to right.
	For example, $\forall x\exists yP(x,y)$ is not equivalent to $\exists y\forall xP(x,y).$ Thus, ordering is important.
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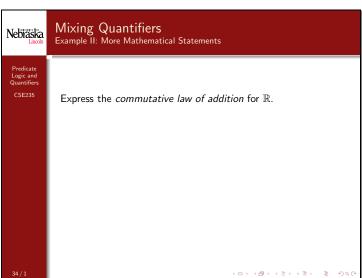


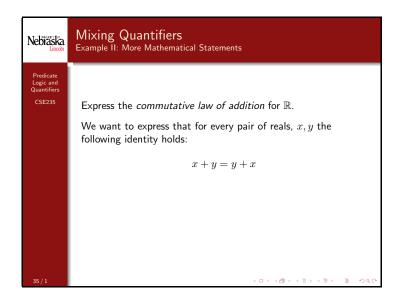


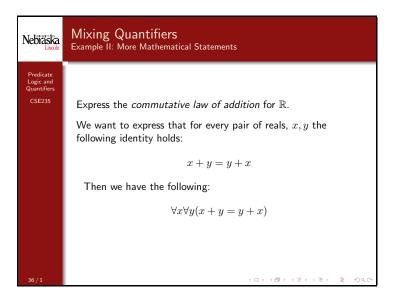


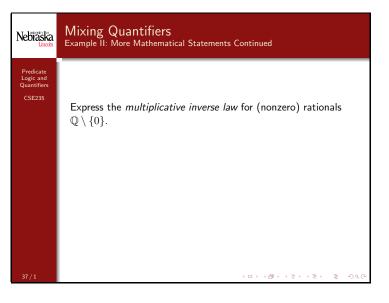


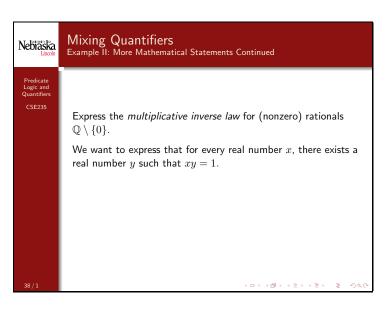


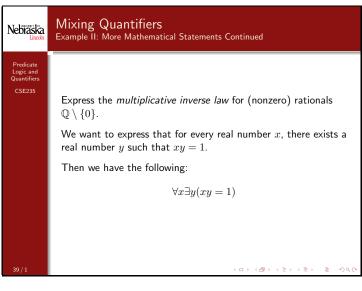


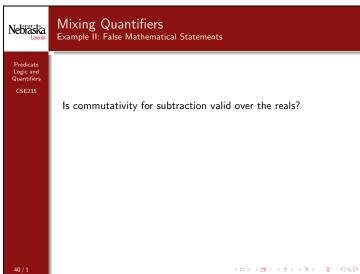


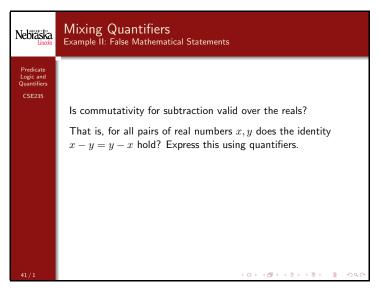


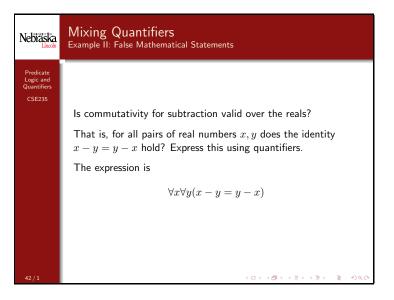


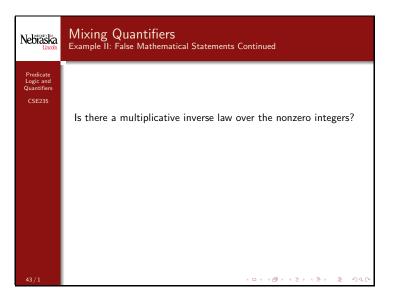


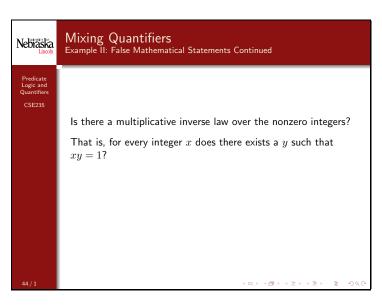


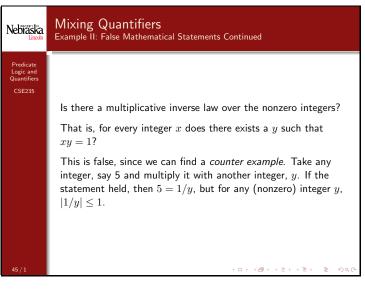


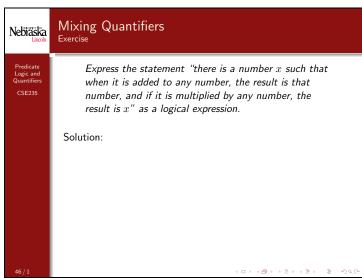


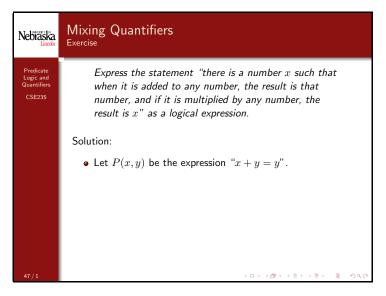


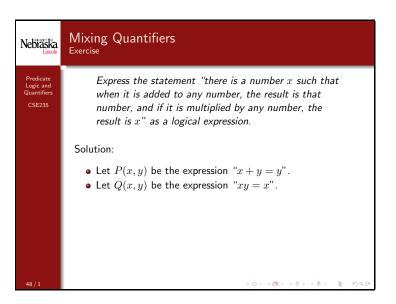














## Mixing Quantifiers

Predicate Logic and Quantifiers

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

#### Solution:

- Let P(x,y) be the expression "x + y = y".
- Let Q(x,y) be the expression "xy = x".
- Then the expression is

$$\exists x \forall y (P(x,y) \land Q(x,y))$$

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## Mixing Quantifiers

Predicate Logic and Quantifiers

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$$\exists x \forall y (P(x,y) \land Q(x,y))$$

 Over what universe(s) of discourse does this statement hold?

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## Mixing Quantifiers Exercise

Predicate Logic and Quantifiers CSE235 Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

#### Solution:

- Let P(x,y) be the expression "x+y=y".
- Let Q(x,y) be the expression "xy = x".
- Then the expression is

$$\exists x \forall y (P(x,y) \land Q(x,y))$$

- Over what universe(s) of discourse does this statement hold?
- This is the additive identity law and holds for  $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$  but does not hold for  $\mathbb{Z}^+$ .



#### Binding Variables I

Predicate Logic and Quantifiers CSE235

When a quantifier is used on a variable x, we say that x is bound. If no quantifier is used on a variable in a predicate statement, it is called *free*.

#### Example

In the expression  $\exists x \forall y P(x,y)$  both x and y are bound. In the expression  $\forall x P(x,y), \ x$  is bound, but y is free.

A statement is called a *well-formed formula*, when all variables are properly quantified.

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#### Binding Variables II

Predicate Logic and Quantifiers

The set of all variables bound by a common quantifier is the *scope* of that quantifier.

#### Example

In the expression  $\exists x,y \forall z P(x,y,z,c)$  the scope of the existential quantifier is  $\{x,y\}$ , the scope of the universal quantifier is just z and c has no scope since it is free.



#### Negation

Logic and Quantifiers

CSE235

Just as we can use negation with propositions, we can use them with quantified expressions.

#### Lemma

Let P(x) be a predicate. Then the following hold.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).

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## Negation

Statement	True When	False When
$\neg \exists x P(x) \equiv$	For every $x$ , $P(x)$ is	There is an $x$ for
$\forall x \neg P(x)$	false.	which $P(x)$ is true.
	There is an $x$ for	P(x) is true for every
$\exists x \neg P(x)$	which $P(x)$ is false.	x.

Table: Truth Values of Negated Quantifiers



#### English into Logic

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Logic is more precise than English.

Transcribing English to Logic and vice versa can be tricky.

When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:

- Use  $\forall$  with  $\Rightarrow$ 
  - $\mathsf{Example:}\ \forall x Lion(x) \Rightarrow Fierce(x)$  $\forall x Lion(x) \land Fierce(x)$  means "everyone is a lion and everyone is fierce"
- Use ∃ with ∧

Example:  $\exists x Lion(x) \land Drinks(x, coffee)$ : holds when you have at least one lion that drinks coffee  $\exists x Lion(x) \Rightarrow Drinks(x, coffee)$  holds when you have people even though no lion drinks coffee.





#### Prolog

Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Calculus. It was developped by the logicians of the artificial intelligence community for symbolic reasoning.

- Prolog allows the user to express facts and rules
- Facts are proposational functions: student(juana), enrolled(juana,cse235), instructor(patel,cse235), etc.
- Rules are implications with conjunctions: teaches(X,Y) := instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as: ?enrolled(juana,cse478)
  - ?enrolled(X,cse478)
  - ?teaches(X,juana)

  - by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5.



#### Conclusion

CSE235

#### Examples? Exercises?

- Rewrite the expression,  $\neg \forall x \big(\exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z)\big)$
- Let P(x,y) denote "x is a factor of y" where  $x \in \{1,2,3,\ldots\}$  and  $y \in \{2,3,4,\ldots\}.$  Let Q(y) denote " $\forall x [P(x,y) \rightarrow ((x=y) \lor (x=1))]$ ". When is Q(y)true?



#### Conclusion

#### Examples? Exercises?

- Rewrite the expression,  $\neg \forall x \big( \exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z) \big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x \big( \forall y \exists z \neg P(x, y, z) \lor \forall z \exists y \neg P(x, y, z) \big)$$

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$$\exists x (\forall y \exists z \neg P(x, y, z) \lor \forall z \exists y \neg P(x, y, z))$$

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- Answer: Only when y is a prime number.



## Extra Question

Predicate Logic and Quantifiers CSE235

Some students wondered if

$$\forall x, y P(x, y) \equiv \forall x P(x, y) \land \forall y P(x, y)$$

This is certainly not true. In the left-hand side, both x and y are bound. In the right-hand side, x is bound in the first predicate, but y is free. In the second predicate, y is bound but x is free.

All variables that occur in a propositional function must be bound to turn it into a proposition.

Thus, the left-hand side is a proposition, but the right-hand side is not. How can they be equivalent?

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