Predicate Logic and Quantifiers

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Sections 1.3–1.4 of Rosen

Introduction

Consider the following statements:

$$x > 3, \quad x = y + 3, \quad x + y = z$$

The truth value of these statements has no meaning without specifying the values of x, y, z.

However, we *can* make propositions out of such statements.

A *predicate* is a property that is affirmed or denied about the *subject* (in logic, we say "variable" or "argument") of a *statement*.

$$\text{“}\underbrace{x}_{\text{subject}}\ \underbrace{\text{is greater than 3”}}_{\text{predicate}}$$

Terminology: affirmed = holds = is true; denied = does not hold = is not true.

Propositional Functions

To write in predicate logic:

$$\text{“}\underbrace{x}_{\text{subject}}\ \underbrace{\text{is greater than 3”}}_{\text{predicate}}$$

We introduce a (functional) symbol for the predicate, and put the subject as an argument (to the functional symbol): $P(x)$

Examples:

- Father(x): unary predicate
- Brother(x,y): binary predicate
- Sum(x,y,z): ternary predicate
- P(x,y,z,t): n-ary predicate

Propositional Functions

Definition

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the *propositional function* P. Here, $(x_1, x_2, \ldots, x_n)$ is an n-tuple and P is a predicate.

You can think of a propositional function as a function that

- Evaluates to true or false.
- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.

Propositional Functions
Example

Example

Let $Q(x, y, z)$ denote the statement "$x^2 + y^2 = z^2$". What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of (x, y, z) make the predicate true?

Propositional Functions
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Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.

Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for (x, y, z) that make this propositional function true—how many right triangles are there?

Universe of Discourse

Consider the previous example. Does it make sense to assign to x the value "blue"?

Intuitively, the *universe of discourse* is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function $P(x) =$ "The test will be on x the 23rd" be?

Universe of Discourse

Multivariate Functions

Moreover, each variable in an n-tuple may have a different universe of discourse.

Let $P(r, g, b, c) =$ "The rgb-value of the color c is (r, g, b)".

For example, $P(255, 0, 0, red)$ is true, while $P(0, 0, 255, green)$ is false.

What are the universes of discourse for (r, g, b, c)?

Quantifiers

Introduction

A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to *quantify* it. That is, the predicate is true (or false) for *all* possible values in the universe of discourse or for *some* value(s) in the universe of discourse.

Such *quantification* can be done with two *quantifiers*: the *universal* quantifier and the *existential* quantifier.

Universal Quantifier

Definition

Definition

The *universal quantification* of a predicate $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of all elements:

$$\forall x P(x) \iff P(n_1) \wedge P(n_2) \wedge \cdots \wedge P(n_k)$$

Universal Quantifier

Example I

- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".
- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

Universal Quantifier

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$$\forall x (Q(x) \rightarrow P(x))$$

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Universal Quantifier
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Predicate Logic and Quantifiers
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- Are these statements true or false?

Universal Quantifier
Example II

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Express the statement "for every x and for every y, $x + y > 10$"

Universal Quantifier
Example II

Predicate Logic and Quantifiers
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Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Universal Quantifier
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Predicate Logic and Quantifiers
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Answer:

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Universal Quantifier
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Predicate Logic and Quantifiers
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Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Note that we can also use the shorthand

$$\forall x, y P(x, y)$$

Existential Quantifier
Definition

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Again, if the universe of discourse is finite, $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of all elements:

$$\exists x P(x) \iff P(n_1) \vee P(n_2) \vee \cdots \vee P(n_k)$$

Existential Quantifier
Example I

Let $P(x, y)$ denote the statement, "$x + y = 5$".

What does the expression,

$$\exists x \exists y P(x)$$

mean?

What universe(s) of discourse make it true?

Existential Quantifier
Example II

Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

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Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

Existential Quantifier
Example II

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Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

The statement can thus be expressed as

$$\exists x P(x)$$

Existential Quantifier
Example II Continued

Question: what is the truth value of $\exists x P(x)$?

Existential Quantifier
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Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such *real* number x can satisfy the predicate.

How can we make it so that it *is* true?

Existential Quantifier
Example II Continued

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How can we make it so that it *is* true?

Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$.

Quantifiers
Truth Values

In general, when are quantified statements true/false?

Statement	True When	False When
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Table: Truth Values of Quantifiers

Mixing Quantifiers I

Existential and universal quantifiers can be used together to quantify a predicate statement; for example,

$$\forall x \exists y P(x, y)$$

is perfectly valid. However, you must be careful—it must be read left to right.

For example, $\forall x \exists y P(x, y)$ is not equivalent to $\exists y \forall x P(x, y)$. Thus, ordering is important.

Mixing Quantifiers II

For example:

- $\forall x \exists y Loves(x, y)$: everybody loves somebody
- $\exists y \forall x Loves(x, y)$: There is someone loved by everyone

Those expressions do not mean the same thing!

Note that $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$, but the converse does not hold

However, you *can* commute *similar* quantifiers; $\exists x \exists y P(x, y)$ is equivalent to $\exists y \exists x P(x, y)$ (which is why our shorthand was valid).

Mixing Quantifiers
Truth Values

Statement	True When	False When
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair, x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table: Truth Values of 2-variate Quantifiers

Mixing Quantifiers
Example I

Express, in predicate logic, the statement that there are an infinite number of integers.

Mixing Quantifiers
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Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Mixing Quantifiers
Example I

Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following.

$$\forall x \exists y P(x, y)$$

Mixing Quantifiers
Example II: More Mathematical Statements

Express the *commutative law of addition* for $\mathbb{R}$.

Mixing Quantifiers
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We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Mixing Quantifiers
Example II: More Mathematical Statements

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We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Then we have the following:

$$\forall x \forall y (x + y = y + x)$$

Mixing Quantifiers
Example II: More Mathematical Statements Continued

Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

Mixing Quantifiers
Example II: More Mathematical Statements Continued

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We want to express that for every real number x, there exists a real number y such that $xy = 1$.

Mixing Quantifiers
Example II: More Mathematical Statements Continued

Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

We want to express that for every real number x, there exists a real number y such that $xy = 1$.

Then we have the following:

$$\forall x \exists y (xy = 1)$$

Mixing Quantifiers
Example II: False Mathematical Statements

Is commutativity for subtraction valid over the reals?

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That is, for all pairs of real numbers x, y does the identity $x - y = y - x$ hold? Express this using quantifiers.

Mixing Quantifiers
Example II: False Mathematical Statements

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That is, for all pairs of real numbers x, y does the identity $x - y = y - x$ hold? Express this using quantifiers.

The expression is

$$\forall x \forall y (x - y = y - x)$$

Mixing Quantifiers
Example II: False Mathematical Statements Continued

Is there a multiplicative inverse law over the nonzero integers?

Mixing Quantifiers
Example II: False Mathematical Statements Continued

Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer x does there exists a y such that $xy = 1$?

Mixing Quantifiers
Example II: False Mathematical Statements Continued

Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer x does there exists a y such that $xy = 1$?

This is false, since we can find a *counter example*. Take any integer, say 5 and multiply it with another integer, y. If the statement held, then $5 = 1/y$, but for any (nonzero) integer y, $|1/y| \leq 1$.

Mixing Quantifiers
Exercise

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

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Exercise

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Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".

Mixing Quantifiers
Exercise

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Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".

Mixing Quantifiers
Exercise

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x,y)$ be the expression "$x+y=y$".
- Let $Q(x,y)$ be the expression "$xy=x$".
- Then the expression is

$$\exists x \forall y \left(P(x,y) \land Q(x,y) \right)$$

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- Over what universe(s) of discourse does this statement hold?

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- Let $Q(x,y)$ be the expression "$xy=x$".
- Then the expression is

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- Over what universe(s) of discourse does this statement hold?
- This is the *additive identity law* and holds for $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$ but does not hold for $\mathbb{Z}^+$.

Binding Variables I

When a quantifier is used on a variable x, we say that x is *bound*. If no quantifier is used on a variable in a predicate statement, it is called *free*.

Example

In the expression $\exists x \forall y P(x,y)$ both x and y are bound.
In the expression $\forall x P(x,y)$, x is bound, but y is free.

A statement is called a *well-formed formula*, when all variables are properly quantified.

Binding Variables II

The set of all variables bound by a common quantifier is the *scope* of that quantifier.

Example

In the expression $\exists x, y \forall z P(x,y,z,c)$ the scope of the existential quantifier is $\{x,y\}$, the scope of the universal quantifier is just z and c has no scope since it is free.

Negation

Just as we can use negation with propositions, we can use them with quantified expressions.

Lemma

Let $P(x)$ be a predicate. Then the following hold.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).

Negation
Truth Values

Statement	True When	False When
$\neg\exists x P(x) \equiv \forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x) \equiv \exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Table: Truth Values of Negated Quantifiers

Prolog

Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Calculus. It was developped by the logicians of the artificial intelligence community for symbolic reasoning.

- Prolog allows the user to express facts and rules
- Facts are proposational functions: student(juana), enrolled(juana,cse235), instructor(patel,cse235), etc.
- Rules are implications with conjunctions:
  teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as:
  ?enrolled(juana,cse478)
  ?enrolled(X,cse478)
  ?teaches(X,juana)
  by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5.

English into Logic

Logic is more precise than English.

Transcribing English to Logic and vice versa can be tricky.

When writing statements with quantifiers, *usually* the correct meaning is conveyed with the following combinations:

- Use $\forall$ with $\Rightarrow$
  Example: $\forall x Lion(x) \Rightarrow Fierce(x)$
  $\forall x Lion(x) \wedge Fierce(x)$ means "everyone is a lion and everyone is fierce"
- Use $\exists$ with $\wedge$
  Example: $\exists x Lion(x) \wedge Drinks(x, coffee)$: holds when you have at least one lion that drinks coffee
  $\exists x Lion(x) \Rightarrow Drinks(x, coffee)$ holds when you have people even though no lion drinks coffee.

Conclusion

Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y \forall z P(x,y,z) \wedge \exists z \forall y P(x,y,z)\big)$
- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \rightarrow ((x=y) \vee (x=1))\big]$". When is $Q(y)$ true?

Conclusion

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- Rewrite the expression, $\neg\forall x\big(\exists y \forall z P(x,y,z) \wedge \exists z \forall y P(x,y,z)\big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x\big(\forall y \exists z \neg P(x,y,z) \vee \forall z \exists y \neg P(x,y,z)\big)$$

- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \rightarrow ((x=y) \vee (x=1))\big]$". When is $Q(y)$ true?

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$$\exists x\big(\forall y \exists z \neg P(x,y,z) \vee \forall z \exists y \neg P(x,y,z)\big)$$

- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \rightarrow ((x=y) \vee (x=1))\big]$". When is $Q(y)$ true?
- Answer: Only when y is a prime number.

Extra Question

Some students wondered if

$$\forall x, y P(x,y) \equiv \forall x P(x,y) \wedge \forall y P(x,y)$$

This is certainly not true. In the left-hand side, both x and y are bound. In the right-hand side, x is bound in the first predicate, but y is free. In the second predicate, y is bound but x is free.

All variables that occur in a propositional function must be bound to turn it into a proposition.

Thus, the left-hand side is a proposition, but the right-hand side is not. How can they be equivalent?