Consider the following statements:

\[ x > 3, \quad x = y + 3, \quad x + y = z \]

The truth value of these statements has no meaning without specifying the values of \( x, y, z \).

However, we can make propositions out of such statements.

A predicate is a property that is affirmed or denied about the subject (in logic, we say "variable" or "argument") of a statement.

"\( x \) is greater than 3"

Terminology: affirmed = holds = is true; denied = does not hold = is not true.

To write in predicate logic:

"\( x \) is greater than 3"

We introduce a (functional) symbol for the predicate, and put the subject as an argument (to the functional symbol): \( P(x) \)

Examples:

- Father(\( x \)): unary predicate
- Brother(\( x, y \)): binary predicate
- Sum(\( x, y, z \)): ternary predicate
- \( P(x, y, z, t) \): \( n \)-ary predicate
Propositional Functions

**Definition**
A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function $P$. Here, $(x_1, x_2, \ldots, x_n)$ is an $n$-tuple and $P$ is a predicate.

You can think of a propositional function as a function that
- Evaluates to true or false.
- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.

**Example**
Let $Q(x, y, z)$ denote the statement "$x^2 + y^2 = z^2$". What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of $(x, y, z)$ make the predicate true?

Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.
Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for $(x, y, z)$ that make this propositional function true—how many right triangles are there?
Consider the previous example. Does it make sense to assign to \( x \) the value "blue"?

Intuitively, the universe of discourse is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function \( P(x) = \text{"The test will be on } x \text{ the 23rd"} \) be?

Moreover, each variable in an \( n \)-tuple may have a different universe of discourse.

Let \( P(r, g, b, c) = \text{"The rgb-value of the color } c \text{ is } (r, g, b)\" \).

For example, \( P(255, 0, 0, \text{red}) \) is true, while \( P(0, 0, 255, \text{green}) \) is false.

What are the universes of discourse for \((r, g, b, c)\)?

A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to quantify it. That is, the predicate is true (or false) for all possible values in the universe of discourse or for some value(s) in the universe of discourse.

Such quantification can be done with two quantifiers: the universal quantifier and the existential quantifier.
Universal Quantifier

Definition

The universal quantification of a predicate \( P(x) \) is the proposition “\( P(x) \) is true for all values of \( x \) in the universe of discourse” We use the notation

\[
\forall x P(x)
\]

which can be read “for all \( x \)”

If the universe of discourse is finite, say \( \{n_1, n_2, \ldots, n_k\} \), then the universal quantifier is simply the conjunction of all elements:

\[
\forall x P(x) \iff P(n_1) \land P(n_2) \land \cdots \land P(n_k)
\]

Example I

Let \( P(x) \) be the predicate “\( x \) must take a discrete mathematics course” and let \( Q(x) \) be the predicate “\( x \) is a computer science student”.

- The universe of discourse for both \( P(x) \) and \( Q(x) \) is all UNL students.
- Express the statement “Every computer science student must take a discrete mathematics course”.

- Express the statement “Everybody must take a discrete mathematics course or be a computer science student”.

\[
\forall x (Q(x) \lor P(x))
\]

Are these statements true or false?
Universal Quantifier

Example I

- Let \( P(x) \) be the predicate “\( x \) must take a discrete mathematics course” and let \( Q(x) \) be the predicate “\( x \) is a computer science student”.
- The universe of discourse for both \( P(x) \) and \( Q(x) \) is all UNL students.
- Express the statement “Every computer science student must take a discrete mathematics course”.
  \[ \forall x (Q(x) \rightarrow P(x)) \]
- Express the statement “Everybody must take a discrete mathematics course or be a computer science student”.
  \[ \forall x (Q(x) \lor P(x)) \]
- Are these statements true or false?

Universal Quantifier

Example II

Express the statement “for every \( x \) and for every \( y, x + y > 10 \)”
Universal Quantifier
Example II

Express the statement “for every $x$ and for every $y$, $x + y > 10$”
Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for $x, y$ is the set of integers.

Answer:
$\forall x \forall y P(x, y)$

Note that we can also use the shorthand
$\forall x, y P(x, y)$
Existential Quantifier

Definition

The **existential quantification** of a predicate \( P(x) \) is the proposition “There exists an \( x \) in the universe of discourse such that \( P(x) \) is true.” We use the notation

\[
\exists x P(x)
\]

which can be read “there exists an \( x \)”

Again, if the universe of discourse is finite, \( \{n_1, n_2, \ldots, n_k\} \), then the existential quantifier is simply the disjunction of all elements:

\[
\exists x P(x) \iff P(n_1) \lor P(n_2) \lor \cdots \lor P(n_k)
\]

Example I

Let \( P(x, y) \) denote the statement, “\( x + y = 5 \).”  
What does the expression,

\[
\exists x \exists y P(x)
\]

mean?  
What universe(s) of discourse make it true?

Example II

Express the statement “there exists a real solution to \( ax^2 + bx - c = 0 \)”
Express the statement “there exists a real solution to $ax^2 + bx - c = 0$”

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the universe of discourse for $x$ is the set of reals. Note here that $a, b, c$ are all fixed constants.

The statement can thus be expressed as

$\exists x P(x)$

Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such real number $x$ can satisfy the predicate.

How can we make it so that it is true?

Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$. 

Notes
Predicate Logic and Quantifiers

CSE235

Existential Quantifier
Example II Continued

Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such real number $x$ can satisfy the predicate.

How can we make it so that it is true?

Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$.

Quantifiers
Truth Values

In general, when are quantified statements true/false?

<table>
<thead>
<tr>
<th>Statement</th>
<th>True When</th>
<th>False When</th>
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<tbody>
<tr>
<td>$\forall x P(x)$</td>
<td>$P(x)$ is true for every $x$.</td>
<td>There is an $x$ for which $P(x)$ is false.</td>
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<td>$\exists x P(x)$</td>
<td>There is an $x$ for which $P(x)$ is true.</td>
<td>$P(x)$ is false for every $x$.</td>
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Table: Truth Values of Quantifiers
Existential and universal quantifiers can be used together to quantify a predicate statement; for example,
\[ \forall x \exists y P(x, y) \]
is perfectly valid. However, you must be careful—it must be read left to right.
For example, \( \forall x \exists y P(x, y) \) is not equivalent to \( \exists y \forall x P(x, y) \). Thus, ordering is important.

For example:
\[ \forall x \exists y \text{Loves}(x, y) \]: everybody loves somebody
\[ \exists y \forall x \text{Loves}(x, y) \]: There is someone loved by everyone
These expressions do not mean the same thing!

Note that \( \exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y) \), but the converse does not hold
However, you can commute similar quantifiers; \( \exists x \exists y P(x, y) \) is equivalent to \( \exists y \exists x P(x, y) \) (which is why our shorthand was valid).

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<tr>
<td>( \forall x \forall y P(x, y) )</td>
<td>( P(x, y) ) is true for every pair ( x, y ).</td>
<td>There is at least one pair, ( x, y ) for which ( P(x, y) ) is false.</td>
</tr>
<tr>
<td>( \forall x \exists y P(x, y) )</td>
<td>For every ( x ), there is a ( y ) for which ( P(x, y) ) is true.</td>
<td>There is an ( x ) for which ( P(x, y) ) is false for every ( y ).</td>
</tr>
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<td>( \exists y \forall x P(x, y) )</td>
<td>There is an ( x ) for which ( P(x, y) ) is true for every ( y ).</td>
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Table: Truth Values of 2-variate Quantifiers
Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following:

$$\forall x \exists y P(x, y)$$
Express the commutative law of addition for \( \mathbb{R} \).

We want to express that for every pair of reals, \( x, y \) the following identity holds:

\[ x + y = y + x \]

Then we have the following:

\[ \forall x \forall y (x + y = y + x) \]
Express the multiplicative inverse law for (nonzero) rationals \( \mathbb{Q} \setminus \{0\} \).

We want to express that for every real number \( x \), there exists a real number \( y \) such that \( xy = 1 \).

Then we have the following:

\[ \forall x \exists y (xy = 1) \]
Is commutativity for subtraction valid over the reals?

That is, for all pairs of real numbers $x, y$ does the identity $x - y = y - x$ hold? Express this using quantifiers.

The expression is

$$\forall x \forall y (x - y = y - x)$$
Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer \( x \) does there exists a \( y \) such that \( xy = 1 \)?

This is false, since we can find a counter example. Take any integer, say 5 and multiply it with another integer, \( y \). If the statement held, then \( 5 = 1/y \), but for any (nonzero) integer \( y \), \( |1/y| \leq 1 \).
Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let P(x, y) be the expression "x + y = y".
- Let Q(x, y) be the expression "xy = x".

Over what universe(s) of discourse does this statement hold?

This is the additive identity law and holds for N, Z, R, Q, but does not hold for Z+.
Express the statement “there is a number $x$ such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is $x$” as a logical expression.

Solution:

- Let $P(x, y)$ be the expression “$x + y = y$”.
- Let $Q(x, y)$ be the expression “$xy = x$”.
- Then the expression is
  \[ \exists x \forall y (P(x, y) \land Q(x, y)) \]

Over what universe(s) of discourse does this statement hold?

This is the additive identity law and holds for $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$ but does not hold for $\mathbb{Z}^+$. 
When a quantifier is used on a variable \( x \), we say that \( x \) is \textit{bound}. If no quantifier is used on a variable in a predicate statement, it is called \textit{free}.

**Example**

In the expression \( \exists x \forall y P(x, y) \) both \( x \) and \( y \) are bound.

In the expression \( \forall x P(x, y) \), \( x \) is bound, but \( y \) is free.

A statement is called a \textit{well-formed formula}, when all variables are properly quantified.

The set of all variables bound by a common quantifier is the \textit{scope} of that quantifier.

**Example**

In the expression \( \exists x, y \forall z P(x, y, z, c) \) the scope of the existential quantifier is \( \{x, y\} \), the scope of the universal quantifier is just \( z \) and \( c \) has no scope since it is free.

Just as we can use negation with propositions, we can use them with quantified expressions.

**Lemma**

Let \( P(x) \) be a predicate. Then the following hold.

\[
\neg \forall x P(x) \equiv \exists x \neg P(x)
\]

\[
\neg \exists x P(x) \equiv \forall x \neg P(x)
\]

This is essentially a quantified version of De Morgan’s Law (in fact if the universe of discourse is finite, it is exactly De Morgan’s law).
Negation

Truth Values

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<td>For every ( x ), ( P(x) ) is false.</td>
<td>There is an ( x ) for which ( P(x) ) is true.</td>
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<tr>
<td>( \neg \forall x P(x) \equiv \exists x \neg P(x) )</td>
<td>There is an ( x ) for which ( P(x) ) is false.</td>
<td>( P(x) ) is true for every ( x ).</td>
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Table: Truth Values of Negated Quantifiers

Prolog

Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Calculus. It was developed by the logicians of the artificial intelligence community for symbolic reasoning.

- Prolog allows the user to express facts and rules
- Facts are propositional functions: student(juana), enrolled(juana,cse235), instructor(patel,cse235), etc.
- Rules are implications with conjunctions: teaches\((X,Y)\) :- instructor\((X,Z), enrolled\((Y,Z)\)
- Prolog answers queries such as:
  - \(?\text{enrolled}(\text{juana}, \text{cse}478)\)
  - \(?\text{enrolled}(\text{X}, \text{cse}478)\)
  - \(?\text{teaches}(\text{X}, \text{juana})\)
- by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5.

English into Logic

Logic is more precise than English.

Transcribing English to Logic and vice versa can be tricky.

When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:

- Use \( \forall \) with \( \Rightarrow \)
  - Example: \( \forall x \text{Lion}(x) \Rightarrow \text{Fierce}(x) \)
  - \( \forall x \text{Lion}(x) \land \text{Fierce}(x) \) means “everyone is a lion and everyone is fierce”
- Use \( \exists \) with \( \land \)
  - Example: \( \exists x \text{Lion}(x) \land \text{Drinks}(x, \text{coffee}) \): holds when you have at least one lion that drinks coffee.
  - \( \exists x \text{Lion}(x) \Rightarrow \text{Drinks}(x, \text{coffee}) \) holds when you have people even though no lion drinks coffee.
Examples? Exercises?

- Rewrite the expression,
  $$\neg \forall x (\exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z))$$

- Let $P(x, y)$ denote "$x$ is a factor of $y$" where $x \in \{1, 2, 3, \ldots\}$ and $y \in \{2, 3, 4, \ldots\}$. Let $Q(y)$ denote "$\forall x [P(x, y) \rightarrow ((x = y) \lor (x = 1))]$". When is $Q(y)$ true?

  Answer: Only when $y$ is a prime number.
Some students wondered if

\[ \forall x, y P(x, y) \equiv \forall x P(x, y) \land \forall y P(x, y) \]

This is certainly not true. In the left-hand side, both \(x\) and \(y\) are bound. In the right-hand side, \(x\) is bound in the first predicate, but \(y\) is free. In the second predicate, \(y\) is bound but \(x\) is free.

*All variables that occur in a propositional function must be bound to turn it into a proposition.*

Thus, the left-hand side is a proposition, but the right-hand side is not. How can they be equivalent?