Number Theory: Applications

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Sections 2.4–2.6 of Rosen

Number Theory: Applications

Results from Number Theory have *countless* applications in mathematics as well as in practical applications including security, memory management, authentication, coding theory, etc. We will only examine (in breadth) a few here.

- Hash Functions
- Pseudorandom Numbers
- Fast Arithmetic Operations
- Cryptography

Hash Functions I

Some notation: $\mathbb{Z}_m = \{0, 1, 2, \ldots, m-2, m-1\}$

Define a *hash function* $h : \mathbb{Z} \to \mathbb{Z}_m$ as

$$h(k) = k \bmod m$$

That is, h maps all integers into a subset of size m by computing the remainder of k/m.

Hash Functions II

In general, a hash function should have the following properties

- It must be easily computable.
- It should distribute items as evenly as possible among all values addresses. To this end, m is usually chosen to be a prime number. It is also common practice to define a hash function that is dependent on each bit of a key
- It must be an onto function (surjective).

Hashing is so useful that many languages have support for hashing (perl, Lisp, Python).

Hash Functions III

However, the function is clearly not one-to-one. When two elements, $x_1 \neq x_2$ *hash* to the same value, we call it a *collision*.

There are many methods to resolve collisions, here are just a few.

- Open Hashing (aka separate chaining) – each hash address is the head of a linked list. When collisions occur, the new key is appended to the end of the list.
- Closed Hashing (aka open addressing) – when collisions occur, we attempt to hash the item into an adjacent hash address. This is known as *linear probing*.

Pseudorandom Numbers

Many applications, such as randomized algorithms, require that we have access to a random source of information (random numbers).

However, there is not *truly random* source in existence, only *weak random sources*: sources that *appear* random, but for which we do not know the probability distribution of events.

Pseudorandom numbers are numbers that are generated from weak random sources such that their distribution is "random enough".

Pseudorandom Numbers I

Linear Congruence Method

One method for generating pseudorandom numbers is the *linear congruential method*.

Choose four integers:

- m, the modulus,
- a, the multiplier,
- c the increment and
- x_0 the seed.

Such that the following hold:

- $2 \leq a < m$
- $0 \leq c < m$
- $0 \leq x_o < m$

Our goal will be to generate a sequence of pseudorandom numbers,

$$\{x_n\}_{n=1}^{\infty}$$

with $0 \leq x_n \leq m$ by using the congruence

$$x_{n+1} = (ax_n + c) \ \mathbf{mod} \ m$$

For certain choices of m, a, c, x_0, the sequence $\{x_n\}$ becomes *periodic*. That is, after a certain point, the sequence begins to repeat. Low periods lead to poor generators.

Furthermore, some choices are better than others; a generator that creates a sequence $0, 5, 0, 5, 0, 5, \ldots$ is obvious bad—its not uniformly distributed.

For these reasons, very large numbers are used in practice.

Example

Let $m = 17, a = 5, c = 2, x_0 = 3$. Then the sequence is as follows.

- $x_{n+1} = (ax_n + c) \textbf{ mod } m$

Linear Congruence Method
Example

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- $x_1 = (5 \cdot x_0 + 2) \textbf{ mod } 17 = 0$

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Linear Congruence Method

Example

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- $x_3 = (5 \cdot x_2 + 2) \textbf{ mod } 17 = 12$

Example

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- $x_{n+1} = (ax_n + c) \mathbf{~mod~} m$
- $x_1 = (5 \cdot x_0 + 2) \mathbf{~mod~} 17 = 0$
- $x_2 = (5 \cdot x_1 + 2) \mathbf{~mod~} 17 = 2$
- $x_3 = (5 \cdot x_2 + 2) \mathbf{~mod~} 17 = 12$
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- $x_6 = (5 \cdot x_5 + 2) \textbf{ mod } 17 = 15$
- $x_7 = (5 \cdot x_6 + 2) \textbf{ mod } 17 = 9$
- $x_8 = (5 \cdot x_7 + 2) \textbf{ mod } 17 = 13$ etc.

Representation of Integers I

This should be old-hat to you, but we review it to be complete (it is also discussed in great detail in your textbook).

Any integer n can be uniquely expressed in any base b by the following expression.

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_2 b^2 + a_1 b + a_0$$

In the expression, each coefficient a_i is an integer between 0 and $b - 1$ inclusive.

Representation of Integers II

For $b = 2$, we have the usual binary representation.
$b = 8$, gives us the octal representation.
$b = 16$ gives us the hexadecimal representation.
$b = 10$ gives us our usual decimal system.

We use the notation

$$(a_k a_{k-1} \cdots a_2 a_1 a_0)_b$$

For $b = 10$, we omit the parentheses and subscript. We also omit leading 0s.

Example

$$\begin{aligned}
(\mathrm{B}9)_{16} &= 11 \cdot 16^1 + 9 \cdot 16^0 \\
&= 176 + 9 = 185 \\
(271)_8 &= 2 \cdot 8^2 + 7 \cdot 8^1 + 1 \cdot 8^0 = 128 + 56 + 1 \\
&= 185 \\
(1011\ 1001)_2 &= 1 \cdot 2^7 + 0 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 \\
&\quad + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 185
\end{aligned}$$

You can verify the following on your own:

$$134 = (1000\ 0110)_2 = (206)_8 = (86)_{16}$$

$$44613 = (1010\ 1110\ 0100\ 0101)_2 = (127105)_8 = (\mathrm{AE}45)_{16}$$

There is a simple and obvious algorithm to compute the base b expansion of an integer.

BASE b EXPANSION

INPUT : A nonnegative integer n and a base b.
OUTPUT : The base b expansion of n.

1. $q \leftarrow n$
2. $k \leftarrow 0$
3. WHILE $q \neq 0$ DO
4. $\quad a_k \leftarrow q \textbf{ mod } b$
5. $\quad q \leftarrow \lfloor \frac{q}{b} \rfloor$
6. $\quad k \leftarrow k + 1$
7. END
8. **output** $(a_{k-1}a_{k-2}\cdots a_1 a_0)$

What is its complexity?

Integer Operations I

You should already know how to add and multiply numbers in binary expansions.

If not, we can go through some examples.

In the textbook, you have 3 algorithms for computing:

1. Addition of two integers in binary expansion; runs in $O(n)$.
2. Product of two integers in binary expansion; runs in $O(n^2)$ (an algorithm that runs in $O(n^{1.585})$ exists).
3. **div** and **mod** for

$$\begin{aligned} q &= a \textbf{ div } d \\ r &= a \textbf{ mod } d \end{aligned}$$

The algorithm runs in $O(q \log a)$ but an algorithm that runs in $O(\log q \log a)$ exists.

Modular Exponentiation I

One useful arithmetic operation that is greatly simplified is modular exponentiation.

Say we want to compute

$$\alpha^n \textbf{ mod } m$$

where n is a *very large* integer. We *could* simply compute

$$\underbrace{\alpha \cdot \alpha \cdot \cdots \cdot \alpha}_{n \text{ times}}$$

We make sure to **mod** each time we multiply to prevent the product from growing too big. This requires $\mathcal{O}(n)$ operations.

Modular Exponentiation II

We can do better. Intuitively, we can perform a *repeated squaring* of the base,

$$\alpha, \alpha^2, \alpha^4, \alpha^8, \ldots$$

requiring $\log n$ operations instead.

Formally, we note that

$$\begin{aligned} \alpha^n &= \alpha^{b_k 2^k + b_{k-1} 2^{k-1} + \cdots + b_1 2 + b_0} \\ &= \alpha^{b_k 2^k} \times \alpha^{b_{k-1} 2^{k-1}} \times \cdots \times \alpha^{2 b_1} \times \alpha^{b_0} \end{aligned}$$

So we can compute α^n by evaluating each term as

$$\alpha^{b_i 2^i} = \begin{cases} \alpha^{2^i} & \text{if } b_i = 1 \\ 1 & \text{if } b_i = 0 \end{cases}$$

Modular Exponentiation III

We can save computation because we can simply square previous values:

$$\alpha^{2^i} = (\alpha^{2^{i-1}})^2$$

We still evaluate each term independently however, since we will need it in the next term (though the accumulated value is only multiplied by 1).

Modular Exponentiation IV

Modular Exponentiation

Input : Integers α, m and $n = (b_k b_{k-1} \ldots b_1 b_0)$ in binary.

Output : α^n **mod** m

term $= \alpha$
if $(b_0 = 1)$ then
    product $= \alpha$
end
else
    product $= 1$
end
for $i = 1 \ldots k$ do
    term $=$ term $\times$ term **mod** m
    if $(b_i = 1)$ then
        product $=$ product $\times$ term **mod** m
    end
end
output product

Example

Compute $12^{26} \bmod 17$ using Modular Exponentiation.

1	1	0	1	0	$= (26)_2$
4	3	2	1	-	i
					term
					product

Example

Compute 12^{26} **mod** 17 using Modular Exponentiation.

1	1	0	1	0	$= (26)_2$
4	3	2	1	-	i
				12	term
				1	product

Example

Compute $12^{26} \bmod 17$ using Modular Exponentiation.

1	1	0	1	0	$= (26)_2$
4	3	2	1	-	i
			8	12	term
			8	1	product

Example

Compute $12^{26} \textbf{ mod } 17$ using Modular Exponentiation.

1	1	0	1	0	$= (26)_2$
4	3	2	1	-	i
		13	8	12	term
		8	8	1	product

Example

Compute $12^{26} \bmod 17$ using Modular Exponentiation.

1	1	0	1	0	$= (26)_2$
4	3	2	1	-	i
	16	13	8	12	term
	9	8	8	1	product

Example

Compute $12^{26} \bmod 17$ using Modular Exponentiation.

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4	3	2	1	-	i
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Example

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1	1	0	1	0	$= (26)_2$
4	3	2	1	-	i
1	16	13	8	12	term
9	9	8	8	1	product

Thus,

$$12^{26} \bmod 17 = 9$$

Euclid's Algorithm

Recall that we can find the $\gcd$ (and thus lcm) by finding the prime factorization of the two integers.

However, the only algorithms known for doing this are exponential (indeed, computer security *depends* on this).

We can, however, compute the $\gcd$ in polynomial time using *Euclid's Algorithm*.

Euclid's Algorithm I

Intuition

Consider finding the $\gcd(184, 1768)$. Dividing the large by the smaller, we get that

$$1768 = 184 \cdot 9 + 112$$

Using algebra, we can reason that any divisor of 184 and 1768 must also be a divisor of the remainder, 112. Thus,

$$\gcd(184, 1768) = \gcd(184, 112)$$

Continuing with our division we eventually get that

$$\begin{aligned} \gcd(184, 1768) &= \gcd(184, 112) \\ &= \gcd(184, 72) \\ &= \gcd(184, 40) \\ &= \gcd(184, 24) \\ &= \gcd(184, 16) \\ &= \gcd(184, 8) = 8 \end{aligned}$$

This concept is formally stated in the following Lemma.

Lemma

Let $a = bq + r$, $a, b, q, r \in \mathbb{Z}$, *then*

$$\gcd(a, b) = \gcd(b, r)$$

The algorithm we present here is actually the *Extended* Euclidean Algorithm. It keeps track of more information to find integers such that the $\gcd$ can be expressed as a *linear combination*.

Theorem

If a and b are positive integers, then there exist integers s, t such that

$$\gcd(a, b) = sa + tb$$

INPUT : Two positive integers a, b.

OUTPUT : $r = \gcd(a, b)$ and s, t such that $sa + tb = \gcd(a, b)$.

$a_0 = a,\ b_0 = b$

$t_0 = 0,\ t = 1$

$s_0 = 1,\ s = 0$

$q = \lfloor \frac{a_0}{b_0} \rfloor$

$r = a_0 - qb_0$

WHILE $r > 0$ DO

    $\mathrm{temp} = t_0 - qt$

    $t_0 = t, t = \mathrm{temp}$

    $\mathrm{temp} = s_0 - qs$

    $s_0 = s, s = \mathrm{temp}$

    $a_0 = b_0,\ b_0 = r$

    $q = \lfloor \frac{a_0}{b_0} \rfloor,\ r = a_0 - qb_0$

    IF $r > 0$ THEN

        $\gcd = r$

    END

END

output $\gcd, s, t$

Algorithm 1: EXTENDEDEUCLIDIANALGORITHM

Euclid's Algorithm
Example

a_0	b_0	t_0	t	s_0	s	q	r

a_0	b_0	t_0	t	s_0	s	q	r
27	58	0	1	1	0	0	27

Euclid's Algorithm
Example

a_0	b_0	t_0	t	s_0	s	q	r
27	58	0	1	1	0	0	27
58	27	1	0	0	1	2	4

Euclid's Algorithm
Example

a_0	b_0	t_0	t	s_0	s	q	r
27	58	0	1	1	0	0	27
58	27	1	0	0	1	2	4
27	4	0	1	1	-2	6	3

a_0	b_0	t_0	t	s_0	s	q	r
27	58	0	1	1	0	0	27
58	27	1	0	0	1	2	4
27	4	0	1	1	-2	6	3
4	3	1	-6	-2	13	1	1

Euclid's Algorithm
Example

a_0	b_0	t_0	t	s_0	s	q	r
27	58	0	1	1	0	0	27
58	27	1	0	0	1	2	4
27	4	0	1	1	-2	6	3
4	3	1	-6	-2	13	1	1
3	1	-6	7	13	-15	3	0

Euclid's Algorithm
Example

a_0	b_0	t_0	t	s_0	s	q	r
27	58	0	1	1	0	0	27
58	27	1	0	0	1	2	4
27	4	0	1	1	-2	6	3
4	3	1	-6	-2	13	1	1
3	1	-6	7	13	-15	3	0

Therefore,

$$\gcd(27, 58) = 1 = (-15)27 + (7)58$$

Example

Compute $\gcd(25480, 26775)$ and find s, t such that

$$\gcd(25480, 26775) = 25480s + 26775t$$

a_0	b_0	t_0	t	s_0	s	q	r
25480	26775	0	1	1	0	0	25480
26775	25480	1	0	0	1	1	1295
25480	1295	0	1	1	-1	19	875
1295	875	1	-19	-1	20	1	420
875	420	-19	20	20	-21	2	35
420	35	20	-59	-21	62	12	0

Therefore,

$$\gcd(25480, 26775) = 35 = (62)25480 + (-59)26775$$

In summary:

- Using the Euclid's Algorithm, we can compute $r = \gcd(a, b)$, where a, b, r are integers.
- Using the Extended Euclide's Algorithm, we can compute the integers r, s, t such that $\gcd(a, b) = r = sa + tb$.

We can use the Extended Euclide's Algorithm to:

- Compute the inverse of an integer a modulo m, where $\gcd(a, m)$=1. (The inverse of a exists and is unique modulo m when $\gcd(a, m)$=1.)
- Solve an equation of linear congruence $ax \equiv b (\bmod m)$, where $\gcd(a, m)$=1

Problem: Compute the inverse of a modulo m with $\gcd(a,m)$=1, that is find a^{-1} such that $a.a^{-1} \equiv 1 (\bmod m)$

$\gcd(a,m) = 1 \Rightarrow 1 = sa + tm$.
Using the EEA, we can find s and t.

$1 = sa + tm \equiv sa (\bmod m) \Rightarrow s = a^{-1}$.

Example: Find the inverse of 5 modulo 9.

Problem: Solve $ax \equiv b(\bmod\ m)$, where $\gcd(a, m){=}1$.

Solution:

- Find a^{-1} the inverse of a module m.
- Multiply the two terms of $ax \equiv b(\bmod\ m)$ by a^{-1}.
  $ax \equiv b(\bmod\ m) \Rightarrow$
  $a^{-1}ax \equiv a^{-1}b(\bmod\ m) \Rightarrow$
  $x \equiv a^{-1}b(\bmod\ m)$.

Example: Solve $5x \equiv 6(\bmod\ 9)$.

Chinese Remainder Theorem

We've already seen an application of linear congruences (pseudorandom number generators).

However, *systems* of linear congruences also have many applications (as we will see).

A system of linear congruences is simply a set of equivalences over a single variable.

Example

$$\begin{aligned} x &\equiv 5 \pmod 2 \\ x &\equiv 1 \pmod 5 \\ x &\equiv 6 \pmod 9 \end{aligned}$$

Chinese Remainder Theorem

Theorem (Chinese Remainder Theorem)

Let $m_1, m_2, \ldots, m_n$ *be pairwise relatively prime positive integers. The system*

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_n \pmod{m_n} \end{aligned}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

How do we *find* such a solution?

This is a good example of a constructive proof; the construction gives us a procedure by which to solve the system. The process is as follows.

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1. Compute $m = m_1 m_2 \cdots m_n$.
2. For each $k = 1, 2, \ldots, n$ compute

$$M_k = \frac{m}{m_k}$$

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1. Compute $m = m_1 m_2 \cdots m_n$.
2. For each $k = 1, 2, \ldots, n$ compute
$$M_k = \frac{m}{m_k}$$
3. For each $k = 1, 2, \ldots, n$ compute the inverse, y_k of $M_k \bmod m_k$ (note these are *guaranteed* to exist by a Theorem in the previous slide set).

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3. For each $k = 1, 2, \ldots, n$ compute the inverse, y_k of $M_k \bmod m_k$ (note these are *guaranteed* to exist by a Theorem in the previous slide set).
4. The solution is the sum
$$x = \sum_{k=1}^{n} a_k M_k y_k$$

Example

Give the unique solution to the system

$$\begin{aligned} x &\equiv 2 \pmod 4 \\ x &\equiv 1 \pmod 5 \\ x &\equiv 6 \pmod 7 \\ x &\equiv 3 \pmod 9 \end{aligned}$$

First, $m = 4 \cdot 5 \cdot 7 \cdot 9 = 1260$ and

$$\begin{aligned} M_1 &= \frac{1260}{4} = 315 \\ M_2 &= \frac{1260}{5} = 252 \\ M_3 &= \frac{1260}{7} = 180 \\ M_4 &= \frac{1260}{9} = 140 \end{aligned}$$

The inverses of each of these is $y_1 = 3, y_2 = 3, y_3 = 3$ and $y_4 = 2$. Therefore, the unique solution is

$$\begin{aligned} x &= a_1M_1y_1 + a_2M_2y_2 + a_3M_3y_3 + a_4M_4y_4 \\ &= 2 \cdot 315 \cdot 3 + 1 \cdot 252 \cdot 3 + 6 \cdot 180 \cdot 3 + 3 \cdot 140 \cdot 2 \\ &= 6726 \mathbf{\ mod\ } 1260 = 426 \end{aligned}$$

To solve the system in the previous example, it was necessary to determine the inverses of M_k modulo m_k—how'd we do that?

One way (as in this case) is to try every single element a, $2 \leq a \leq m-1$ to see if

$$aM_k \equiv 1 \pmod{m}$$

But there is a more efficient way that we already know how to do—*Euclid's Algorithm!*

Computing Inverses

Lemma

Let a, b be relatively prime. Then the linear combination computed by the Extended Euclidean Algorithm,

$$\gcd(a, b) = sa + tb$$

gives the inverse of a modulo b; i.e. $s = a^{-1}$ modulo b.

Note that $t = b^{-1}$ modulo a.

Also note that it may be necessary to take the modulo of the result.

Chinese Remainder Representations

In many applications, it is necessary to perform simple arithmetic operations on *very* large integers.

Such operations become inefficient if we perform them bitwise.

Instead, we can use *Chinese Remainder Representations* to perform arithmetic operations of large integers using *smaller* integers saving computations. Once operations have been performed, we can uniquely recover the large integer result.

Lemma

Let $m_1, m_2, \ldots, m_n$ *be pairwise relatively prime integers,* $m_i \geq 2$. *Let*

$$m = m_1 m_2 \cdots m_n$$

Then every integer $a, 0 \leq a < m$ *can be uniquely represented by* n *remainders over* m_i*; i.e.*

$$(a \textbf{ mod } m_1, a \textbf{ mod } m_2, \ldots, a \textbf{ mod } m_n)$$

Example

Let $m_1 = 47, m_2 = 48, m_3 = 49, m_4 = 53$. Compute $2,459,123 + 789,123$ using Chinese Remainder Representations.

By the previous lemma, we can represent any integer up to 5,858,832 by four integers all less than 53.

First,

$$\begin{aligned} 2,459,123 \textbf{ mod } 47 &= 36 \\ 2,459,123 \textbf{ mod } 48 &= 35 \\ 2,459,123 \textbf{ mod } 49 &= 9 \\ 2,459,123 \textbf{ mod } 53 &= 29 \end{aligned}$$

Chinese Remainder Representations II

Example

Next,

$$\begin{aligned} 789,123 \textbf{ mod } 47 &= 40 \\ 789,123 \textbf{ mod } 48 &= 3 \\ 789,123 \textbf{ mod } 49 &= 27 \\ 789,123 \textbf{ mod } 53 &= 6 \end{aligned}$$

So we've reduced our calculations to computing (coordinate wise) the addition:

$$\begin{aligned} (36,35,9,29)+(40,3,27,6) &= (76,38,36,35) \\ &= (29,38,36,35) \end{aligned}$$

Chinese Remainder Representations III
Example

Now we wish to recover the result, so we solve the system of linear congruences,

$$\begin{aligned} x &\equiv 29 \pmod{47} \\ x &\equiv 38 \pmod{48} \\ x &\equiv 36 \pmod{49} \\ x &\equiv 35 \pmod{53} \end{aligned}$$

$$\begin{aligned} M_1 &= 124656 \\ M_2 &= 122059 \\ M_3 &= 119568 \\ M_4 &= 110544 \end{aligned}$$

Chinese Remainder Representations IV
Example

We use the Extended Euclidean Algorithm to find the inverses of each of these w.r.t. the appropriate modulus:

$$\begin{aligned} y_1 &= 4 \\ y_2 &= 19 \\ y_3 &= 43 \\ y_4 &= 34 \end{aligned}$$

And so we have that

$$
\begin{aligned}
x &= 29(124656 \textbf{ mod } 47)4 + 38(122059 \textbf{ mod } 48)19 + \\
&\quad 36(119568 \textbf{ mod } 49)43 + 35(110544 \textbf{ mod } 53)34 \\
&= 3,248,246 \\
&= 2,459,123 + 789,123
\end{aligned}
$$

Caesar Cipher I

Cryptography is the study of secure communication via *encryption*.

One of the earliest uses was in ancient Rome and involved what is now known as a *Caesar cipher*.

This simple encryption system involves a *shift* of letters in a fixed alphabet. Encryption and decryption is simple modular arithmetic.

Caesar Cipher II

In general, we fix an alphabet, Σ and let $m = |\Sigma|$. Second, we fix an secret *key*, an integer k such that $0 < k < m$. Then the encryption and decryption functions are

$$\begin{aligned} e_k(x) &= (x + k) \textbf{ mod } m \\ d_k(y) &= (y - k) \textbf{ mod } m \end{aligned}$$

respectively.

Cryptographic functions must be one-to-one (why?). It is left as an exercise to verify that this Caesar cipher satisfies this condition.

Example

Let $\Sigma = \{A, B, C, \ldots, Z\}$ so $m = 26$. Choose $k = 7$. Encrypt "HANK" and decrypt "KLHU".

"HANK" can be encoded (7-0-13-10), so

$$\begin{aligned}
e(7) &= (7+7) \bmod 26 &= 14 \\
e(0) &= (0+7) \bmod 26 &= 7 \\
e(13) &= (13+7) \bmod 26 &= 20 \\
e(10) &= (10+7) \bmod 26 &= 17
\end{aligned}$$

so the encrypted word is "OHUR".

"KLHU" is encoded as (10-11-7-20), so

Caesar Cipher
Example Continued

"KLHU" is encoded as (10-11-7-20), so

$$e(10) \quad = \quad (10 - 7) \ \mathbf{mod}\ 26 \quad = 3$$

Caesar Cipher
Example Continued

"KLHU" is encoded as (10-11-7-20), so

$$\begin{aligned} e(10) &= (10-7) \textbf{ mod } 26 &= 3 \\ e(11) &= (11-7) \textbf{ mod } 26 &= 4 \end{aligned}$$

"KLHU" is encoded as (10-11-7-20), so

$$\begin{aligned} e(10) &= (10-7) \textbf{ mod } 26 &= 3 \\ e(11) &= (11-7) \textbf{ mod } 26 &= 4 \\ e(7) &= (7-7) \textbf{ mod } 26 &= 0 \end{aligned}$$

"KLHU" is encoded as (10-11-7-20), so

$$\begin{aligned} e(10) &= (10-7) \textbf{ mod } 26 &= 3 \\ e(11) &= (11-7) \textbf{ mod } 26 &= 4 \\ e(7) &= (7-7) \textbf{ mod } 26 &= 0 \\ e(20) &= (20-7) \textbf{ mod } 26 &= 13 \end{aligned}$$

Caesar Cipher
Example Continued

"KLHU" is encoded as (10-11-7-20), so

$$
\begin{aligned}
e(10) &= (10-7) \textbf{ mod } 26 &= 3\\
e(11) &= (11-7) \textbf{ mod } 26 &= 4\\
e(7) &= (7-7) \textbf{ mod } 26 &= 0\\
e(20) &= (20-7) \textbf{ mod } 26 &= 13
\end{aligned}
$$

So the decrypted word is "DEAN".

Affine Cipher I

Clearly, the Caesar cipher is insecure—the key space is only as large as the alphabet.

An alternative (though still not secure) is what is known as an *affine* cipher. Here the encryption and decryption functions are as follows.

$$\begin{aligned} e_k(x) &= (ax + b) \bmod m \\ d_k(y) &= a^{-1}(y - b) \bmod m \end{aligned}$$

Question: How big is the key space?

Example

To ensure a bijection, we choose $m = 29$ to be a prime (why?). Let $a = 10, b = 14$. Encrypt the word "PROOF" and decrypt the message "OBGJLK".

"PROOF" can be encoded as (16-18-15-15-6). The encryption is as follows.

Example

To ensure a bijection, we choose $m = 29$ to be a prime (why?). Let $a = 10, b = 14$. Encrypt the word "PROOF" and decrypt the message "OBGJLK".

"PROOF" can be encoded as (16-18-15-15-6). The encryption is as follows.

$$e(16) \quad = \quad (10 \cdot 16 + 14) \text{ mod } 29 \quad = 0$$

Example

To ensure a bijection, we choose $m = 29$ to be a prime (why?). Let $a = 10, b = 14$. Encrypt the word "PROOF" and decrypt the message "OBGJLK".

"PROOF" can be encoded as (16-18-15-15-6). The encryption is as follows.

$$\begin{aligned} e(16) &= (10 \cdot 16 + 14) \textbf{ mod } 29 &= 0 \\ e(18) &= (10 \cdot 18 + 14) \textbf{ mod } 29 &= 20 \end{aligned}$$

Example

To ensure a bijection, we choose $m = 29$ to be a prime (why?). Let $a = 10, b = 14$. Encrypt the word "PROOF" and decrypt the message "OBGJLK".

"PROOF" can be encoded as (16-18-15-15-6). The encryption is as follows.

$$
\begin{aligned}
e(16) &= (10 \cdot 16 + 14) \ \mathbf{mod}\ 29 &= 0 \\
e(18) &= (10 \cdot 18 + 14) \ \mathbf{mod}\ 29 &= 20 \\
e(15) &= (10 \cdot 15 + 14) \ \mathbf{mod}\ 29 &= 19
\end{aligned}
$$

Example

To ensure a bijection, we choose $m = 29$ to be a prime (why?). Let $a = 10, b = 14$. Encrypt the word "PROOF" and decrypt the message "OBGJLK".

"PROOF" can be encoded as (16-18-15-15-6). The encryption is as follows.

$$\begin{aligned} e(16) &= (10 \cdot 16 + 14) \textbf{ mod } 29 &= 0 \\ e(18) &= (10 \cdot 18 + 14) \textbf{ mod } 29 &= 20 \\ e(15) &= (10 \cdot 15 + 14) \textbf{ mod } 29 &= 19 \\ e(15) &= (10 \cdot 15 + 14) \textbf{ mod } 29 &= 19 \end{aligned}$$

Example

To ensure a bijection, we choose $m = 29$ to be a prime (why?). Let $a = 10, b = 14$. Encrypt the word "PROOF" and decrypt the message "OBGJLK".

"PROOF" can be encoded as (16-18-15-15-6). The encryption is as follows.

$$
\begin{aligned}
e(16) &= (10 \cdot 16 + 14) \textbf{ mod } 29 &= 0 \\
e(18) &= (10 \cdot 18 + 14) \textbf{ mod } 29 &= 20 \\
e(15) &= (10 \cdot 15 + 14) \textbf{ mod } 29 &= 19 \\
e(15) &= (10 \cdot 15 + 14) \textbf{ mod } 29 &= 19 \\
e(6) &= (10 \cdot 6 + 14) \textbf{ mod } 29 &= 16
\end{aligned}
$$

Example

To ensure a bijection, we choose $m = 29$ to be a prime (why?). Let $a = 10, b = 14$. Encrypt the word "PROOF" and decrypt the message "OBGJLK".

"PROOF" can be encoded as (16-18-15-15-6). The encryption is as follows.

$$
\begin{aligned}
e(16) &= (10 \cdot 16 + 14) \bmod 29 &&= 0 \\
e(18) &= (10 \cdot 18 + 14) \bmod 29 &&= 20 \\
e(15) &= (10 \cdot 15 + 14) \bmod 29 &&= 19 \\
e(15) &= (10 \cdot 15 + 14) \bmod 29 &&= 19 \\
e(6) &= (10 \cdot 6 + 14) \bmod 29 &&= 16
\end{aligned}
$$

The encrypted message is "AUPPG".

Affine Cipher

Example Continued

When do we attack? Computing the inverse, we find that $a^{-1} = 3$.

We can decrypt the message "OBGJLK" (14-1-6-9-11-10) as follows.

Affine Cipher
Example Continued

When do we attack? Computing the inverse, we find that $a^{-1} = 3$.

We can decrypt the message "OBGJLK" (14-1-6-9-11-10) as follows.

$$e(14) \;=\; 3(14-14) \mathbf{\ mod\ } 29 \quad = 0 \qquad = A$$

When do we attack? Computing the inverse, we find that $a^{-1} = 3$.

We can decrypt the message "OBGJLK" (14-1-6-9-11-10) as follows.

$$\begin{array}{lcllll} e(14) & = & 3(14-14) \textbf{ mod } 29 & = 0 & = A \\ e(1) & = & 3(1-14) \textbf{ mod } 29 & = 19 & = T \end{array}$$

When do we attack? Computing the inverse, we find that $a^{-1} = 3$.

We can decrypt the message "OBGJLK" (14-1-6-9-11-10) as follows.

$$
\begin{aligned}
e(14) &= 3(14-14) \textbf{ mod } 29 &&= 0 &&= A \\
e(1) &= 3(1-14) \textbf{ mod } 29 &&= 19 &&= T \\
e(6) &= 3(6-14) \textbf{ mod } 29 &&= 5 &&= F
\end{aligned}
$$

Affine Cipher
Example Continued

When do we attack? Computing the inverse, we find that $a^{-1} = 3$.

We can decrypt the message “OBGJLK” (14-1-6-9-11-10) as follows.

$$\begin{array}{lcll} e(14) & = & 3(14-14) \textbf{ mod } 29 & = 0 & = A \\ e(1) & = & 3(1-14) \textbf{ mod } 29 & = 19 & = T \\ e(6) & = & 3(6-14) \textbf{ mod } 29 & = 5 & = F \\ e(9) & = & 3(9-14) \textbf{ mod } 29 & = 14 & = O \end{array}$$

Affine Cipher
Example Continued

When do we attack? Computing the inverse, we find that $a^{-1} = 3$.

We can decrypt the message "OBGJLK" (14-1-6-9-11-10) as follows.

$$\begin{array}{lcllr}
e(14) & = & 3(14-14) \textbf{ mod } 29 & = 0 & = A \\
e(1) & = & 3(1-14) \textbf{ mod } 29 & = 19 & = T \\
e(6) & = & 3(6-14) \textbf{ mod } 29 & = 5 & = F \\
e(9) & = & 3(9-14) \textbf{ mod } 29 & = 14 & = O \\
e(11) & = & 3(11-14) \textbf{ mod } 29 & = 20 & = U
\end{array}$$

Affine Cipher
Example Continued

When do we attack? Computing the inverse, we find that $a^{-1} = 3$.

We can decrypt the message “OBGJLK” (14-1-6-9-11-10) as follows.

$$\begin{aligned}
e(14) &= 3(14-14) \ \mathbf{mod}\ 29 &= 0 &= A \\
e(1) &= 3(1-14) \ \mathbf{mod}\ 29 &= 19 &= T \\
e(6) &= 3(6-14) \ \mathbf{mod}\ 29 &= 5 &= F \\
e(9) &= 3(9-14) \ \mathbf{mod}\ 29 &= 14 &= O \\
e(11) &= 3(11-14) \ \mathbf{mod}\ 29 &= 20 &= U \\
e(10) &= 3(10-14) \ \mathbf{mod}\ 29 &= 17 &= R
\end{aligned}$$

Public-Key Cryptography I

The problem with the Caesar & Affine ciphers (aside from the fact that they are insecure) is that you still need a secure way to exchange the keys in order to communicate.

Public key cryptosystems solve this problem.

- One can publish a *public key*.
- Anyone can encrypt messages.
- However, decryption is done with a *private* key.
- The system is secure if no one can *feasibly* derive the private key from the public one.
- Essentially, encryption should be computationally easy, while decryption should be computationally hard (without the private key).
- Such protocols use what are called "trap-door functions".

Public-Key Cryptography II

Many public key cryptosystems have been developed based on the (assumed) hardness of *integer factorization* and the *discrete log* problems.

Systems such as the *Diffie-Hellman* key exchange protocol (used in SSL, SSH, https) and the *RSA* cryptosystem are the basis of modern secure computer communication.

The RSA Cryptosystem I

The RSA system works as follows.

- Choose 2 (large) primes p, q.
- Compute $n = pq$.
- Compute $\phi(n) = (p-1)(q-1)$.
- Choose a, $2 \leq a \leq \phi(n)$ such that $\gcd(a, \phi(n)) = 1$.
- Compute $b = a^{-1}$ modulo $\phi(n)$.
- Note that a must be relatively prime to $\phi(n)$.
- Publish n, a
- Keep p, q, b private.

The RSA Cryptosystem II

Then the encryption function is simply

$$e_k(x) = x^a \textbf{ mod } n$$

The decryption function is

$$d_k(y) = y^b \textbf{ mod } n$$

Recall that we can compute inverses using the Extended Euclidean Algorithm.

With RSA we want to find $b = a^{-1} \textbf{ mod } \phi(n)$. Thus, we compute

$$\gcd(a, \phi(n)) = sa + t\phi(n)$$

and so $b = s = a^{-1}$ modulo $\phi(n)$.

Example

Let $p = 13, q = 17,\ a = 47$.

We have

- $n = 13 \cdot 17 = 221$.
- $\phi(n) = 12 \cdot 16 = 192$.
- Using the Euclidean Algorithm, $b = 47^{-1} = 143$ modulo $\phi(n)$

$$e(130) = 130^{47} \mathbf{\ mod\ } 221 =$$

$$d(99) = 99^{143} \mathbf{\ mod\ } 221 =$$

Example

Let $p = 13, q = 17$, $a = 47$.

We have

- $n = 13 \cdot 17 = 221$.
- $\phi(n) = 12 \cdot 16 = 192$.
- Using the Euclidean Algorithm, $b = 47^{-1} = 143$ modulo $\phi(n)$

$$e(130) = 130^{47} \ \mathbf{mod}\ 221 = 65$$

$$d(99) = 99^{143} \ \mathbf{mod}\ 221 =$$

Example

Let $p = 13, q = 17$, $a = 47$.

We have

- $n = 13 \cdot 17 = 221$.
- $\phi(n) = 12 \cdot 16 = 192$.
- Using the Euclidean Algorithm, $b = 47^{-1} = 143$ modulo $\phi(n)$

$$e(130) = 130^{47} \mathbf{\ mod\ } 221 = 65$$

$$d(99) = 99^{143} \mathbf{\ mod\ } 221 = 96$$

How can we break an RSA protocol? "Simple"—just factor n.

If we have the two factors p and q, we can easily compute $\phi(n)$ and since we already have a, we can also easily compute $b = a^{-1}$ modulo $\phi(n)$.

Thus, the security of RSA is contingent on the hardness of *integer factorization*.

If someone were to come up with a polynomial time algorithm for factorization (or build a feasible quantum computer and use Shor's Algorithm), breaking RSA may be a trivial matter. Though this is not likely.

In practice, large integers, as big as 1024 bits are used. 2048 bit integers are considered unbreakable by today's computer; 4096 bit numbers are used by the truly paranoid.

But if you care to try, RSA Labs has a challenge:

http://www.rsasecurity.com/rsalabs/node.asp?id=2091

Example

Let $a = 2367$ and let $n = 3127$. Decrypt the message, 1125-2960-0643-0325-1884 (Who is the father of modern computer science?)

Factoring n, we find that $n = 53 \cdot 59$ so

$$\phi(n) = 52 \cdot 58 = 3016$$

Using the Euclidean algorithm, $b = a^{-1} = 79$. Thus, the decryption function is

$$d(x) = x^{79} \mathbf{\ mod\ } 3127$$

Decrypting the message we get that

Using the Euclidean algorithm, $b = a^{-1} = 79$. Thus, the decryption function is

$$d(x) = x^{79} \textbf{ mod } 3127$$

Decrypting the message we get that

$$d(1225) = 1225^{79} \textbf{ mod } 3127 = 112$$

Using the Euclidean algorithm, $b = a^{-1} = 79$. Thus, the decryption function is

$$d(x) = x^{79} \mathbf{~mod~} 3127$$

Decrypting the message we get that

$$\begin{aligned} d(1225) &= 1225^{79} \mathbf{~mod~} 3127 &= 112 \\ d(2960) &= 2960^{79} \mathbf{~mod~} 3127 &= 114 \end{aligned}$$

Using the Euclidean algorithm, $b = a^{-1} = 79$. Thus, the decryption function is

$$d(x) = x^{79} \textbf{ mod } 3127$$

Decrypting the message we get that

$$\begin{aligned} d(1225) &= 1225^{79} \textbf{ mod } 3127 &= 112 \\ d(2960) &= 2960^{79} \textbf{ mod } 3127 &= 114 \\ d(0643) &= 643^{79} \textbf{ mod } 3127 &= 2021 \end{aligned}$$

Using the Euclidean algorithm, $b = a^{-1} = 79$. Thus, the decryption function is

$$d(x) = x^{79} \textbf{ mod } 3127$$

Decrypting the message we get that

$$\begin{aligned}
d(1225) &= 1225^{79} \textbf{ mod } 3127 &= 112 \\
d(2960) &= 2960^{79} \textbf{ mod } 3127 &= 114 \\
d(0643) &= 643^{79} \textbf{ mod } 3127 &= 2021 \\
d(0325) &= 325^{79} \textbf{ mod } 3127 &= 1809
\end{aligned}$$

Using the Euclidean algorithm, $b = a^{-1} = 79$. Thus, the decryption function is

$$d(x) = x^{79} \ \mathbf{mod}\ 3127$$

Decrypting the message we get that

$$\begin{aligned}
d(1225) &= 1225^{79} \ \mathbf{mod}\ 3127 &= 112\\
d(2960) &= 2960^{79} \ \mathbf{mod}\ 3127 &= 114\\
d(0643) &= 643^{79} \ \mathbf{mod}\ 3127 &= 2021\\
d(0325) &= 325^{79} \ \mathbf{mod}\ 3127 &= 1809\\
d(1884) &= 1884^{79} \ \mathbf{mod}\ 3127 &= 1407
\end{aligned}$$

Using the Euclidean algorithm, $b = a^{-1} = 79$. Thus, the decryption function is

$$d(x) = x^{79} \textbf{ mod } 3127$$

Decrypting the message we get that

$$\begin{aligned}
d(1225) &= 1225^{79} \textbf{ mod } 3127 &= 112 \\
d(2960) &= 2960^{79} \textbf{ mod } 3127 &= 114 \\
d(0643) &= 643^{79} \textbf{ mod } 3127 &= 2021 \\
d(0325) &= 325^{79} \textbf{ mod } 3127 &= 1809 \\
d(1884) &= 1884^{79} \textbf{ mod } 3127 &= 1407
\end{aligned}$$

Thus, the message is "ALAN TURING".