Master Theorem

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Introduction to Discrete Mathematics
When analyzing algorithms, recall that we only care about the *asymptotic behavior*.

Recursive algorithms are no different. Rather than *solve* exactly the recurrence relation associated with the cost of an algorithm, it is enough to give an asymptotic characterization.

The main tool for doing this is the *master theorem*. 
Theorem (Master Theorem)

Let $T(n)$ be a monotonically increasing function that satisfies

$$
T(n) = aT\left(\frac{n}{b}\right) + f(n)
\quad T(1) = c
$$

where $a \geq 1$, $b \geq 2$, $c > 0$. If $f(n) \in \Theta(n^d)$ where $d \geq 0$, then

$$
T(n) = \begin{cases} 
\Theta(n^d) & \text{if } a < b^d \\
\Theta(n^d \log n) & \text{if } a = b^d \\
\Theta(n^{\log_b a}) & \text{if } a > b^d 
\end{cases}
$$
You cannot use the Master Theorem if

- $T(n)$ is not monotone, ex: $T(n) = \sin n$
- $f(n)$ is not a polynomial, ex: $T(n) = 2T\left(\frac{n}{2}\right) + 2^n$
- $b$ cannot be expressed as a constant, ex: $T(n) = T(\sqrt{n})$

Note here, that the Master Theorem does not solve a recurrence relation.

Does the base case remain a concern?
Master Theorem

Example 1

Let \( T(n) = T \left( \frac{n}{2} \right) + \frac{1}{2}n^2 + n \). What are the parameters?

\[
\begin{align*}
  a &= \\
  b &= \\
  d &= 
\end{align*}
\]

Therefore which condition?
Master Theorem

Example 1

Let $T(n) = T\left(\frac{n}{2}\right) + \frac{1}{2} n^2 + n$. What are the parameters?

\[
\begin{align*}
    a &= 1 \\
    b &= \\
    d &= 
\end{align*}
\]

Therefore which condition?
Master Theorem

Example 1

Let \( T(n) = T\left(\frac{n}{2}\right) + \frac{1}{2}n^2 + n \). What are the parameters?

\[
\begin{align*}
    a &= 1 \\
    b &= 2 \\
    d &= \\
\end{align*}
\]

Therefore which condition?
Master Theorem

Example 1

Let $T(n) = T\left(\frac{n}{2}\right) + \frac{1}{2}n^2 + n$. What are the parameters?

$$a = 1$$
$$b = 2$$
$$d = 2$$

Therefore which condition?
Master Theorem
Example 1

Let \( T(n) = T \left( \frac{n}{2} \right) + \frac{1}{2} n^2 + n \). What are the parameters?

\[
\begin{align*}
    a &= 1 \\
    b &= 2 \\
    d &= 2 \\
\end{align*}
\]

Therefore which condition?

Since \( 1 < 2^2 \), case 1 applies.
Let $T(n) = T\left(\frac{n}{2}\right) + \frac{1}{2} n^2 + n$. What are the parameters?

\[
\begin{align*}
a &= 1 \\
b &= 2 \\
d &= 2
\end{align*}
\]

Therefore which condition?

Since $1 < 2^2$, case 1 applies.

Thus we conclude that

\[T(n) \in \Theta(n^d) = \Theta(n^2)\]
Let $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42$. What are the parameters?

\[
\begin{align*}
    a &= \quad \quad \\
    b &= \quad \quad \\
    d &= \quad \quad 
\end{align*}
\]

Therefore which condition?
Master Theorem

Example 2

Let \( T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42 \). What are the parameters?

\[
\begin{align*}
a &= 2 \\
b &= \\
d &= 
\end{align*}
\]

Therefore which condition?
Master Theorem
Example 2

Let $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42$. What are the parameters?

\begin{align*}
    a &= 2 \\
    b &= 4 \\
    d &= \\
\end{align*}

Therefore which condition?
Let $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42$. What are the parameters?

$$a = 2$$
$$b = 4$$
$$d = \frac{1}{2}$$

Therefore which condition?
Let $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42$. What are the parameters?

\[
a = 2 \\
b = 4 \\
d = \frac{1}{2}
\]

Therefore which condition?

Since $2 = 4^{\frac{1}{2}}$, case 2 applies.
Master Theorem

Example 2

Let \( T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42 \). What are the parameters?

\[
\begin{align*}
  a &= 2 \\
  b &= 4 \\
  d &= \frac{1}{2}
\end{align*}
\]

Therefore which condition?

Since \( 2 = 4^{\frac{1}{2}} \), case 2 applies.

Thus we conclude that

\[
T(n) \in \Theta(n^d \log n) = \Theta(\sqrt{n} \log n)
\]
Master Theorem

Example 3

Let \( T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1 \). What are the parameters?

\[
\begin{align*}
a &= \\
b &= \\
d &= \\
\end{align*}
\]

Therefore which condition?
Let $T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1$. What are the parameters?

\[
\begin{align*}
a &= 3 \\
\quad b &= \\
\quad d &= 
\end{align*}
\]

Therefore which condition?
Let $T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1$. What are the parameters?

\begin{align*}
    a &= 3 \\
    b &= 2 \\
    d &= 1
\end{align*}

Therefore which condition?

Since $3 > \frac{3}{4}$, case 3 applies. Thus we conclude that $T(n) \in \Theta(n \log_2 3)$, where $\log_2 3 \approx 1.5849$. Can we say that $T(n) \in \Theta(n^{1.5849})$?
Master Theorem
Example 3

Let \( T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1 \). What are the parameters?

\[
\begin{align*}
    a &= 3 \\
    b &= 2 \\
    d &= 1
\end{align*}
\]

Therefore which condition?
Let $T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1$. What are the parameters?

$$a = 3$$
$$b = 2$$
$$d = 1$$

Therefore which condition?

Since $3 > 2^1$, case 3 applies.
Master Theorem
Example 3

Let \( T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1 \). What are the parameters?

\[
\begin{align*}
  a &= 3 \\
  b &= 2 \\
  d &= 1
\end{align*}
\]

Therefore which condition?

Since \( 3 > 2^1 \), case 3 applies. Thus we conclude that

\[
T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})
\]
Let $T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1$. What are the parameters?

\[
\begin{align*}
a &= 3 \\
b &= 2 \\
d &= 1 \\
\end{align*}
\]

Therefore which condition?

Since $3 > 2^1$, case 3 applies. Thus we conclude that

\[
T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})
\]

Note that $\log_2 3 \approx 1.5849$ ... Can we say that

$T(n) \in \Theta(n^{1.5849})$?
Recall that we cannot use the Master Theorem if $f(n)$ (the non-recursive cost) is not polynomial.

There is a limited 4-th condition of the Master Theorem that allows us to consider polylogarithmic functions.

**Corollary**

If $f(n) \in \Theta(n^{\log_b a \log^k n})$ for some $k \geq 0$ then

$$T(n) \in \Theta(n^{\log_b a \log^{k+1} n})$$

This final condition is fairly limited and we present it merely for completeness.
“Fourth” Condition
Example

Say that we have the following recurrence relation:

\[ T(n) = 2T\left(\frac{n}{2}\right) + n \log n \]

Clearly, \( a = 2 \), \( b = 2 \) but \( f(n) \) is not a polynomial. However,

\[ f(n) \in \Theta(n \log n) \]

for \( k = 1 \), therefore, by the 4-th case of the Master Theorem we can say that

\[ T(n) \in \Theta(n \log^2 n) \]