

Introduction to Logic CSE235

Introduction to Logic

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Computer Science & Engineering 235 Introduction to Discrete Mathematics Sections 1.1-1.2 of Rosen and the section of Rose



Introduction I

Introduction to Logic CSE235

Propositional calculus (or logic) is the study of the logical relationship between objects called propositions and forms the basis of all mathematical reasoning.

Definition

A proposition is a statement that is either *true* or *false*, but not both (we usually denote a proposition by letters; p, q, r, s, \ldots).

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Introduction II

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Definition

The value of a proposition is called its *truth value*; denoted by T or 1 if it is true and F or 0 if it is false.

Opinions, interrogative and imperative sentences are not propositions.

Truth table:



Examples I

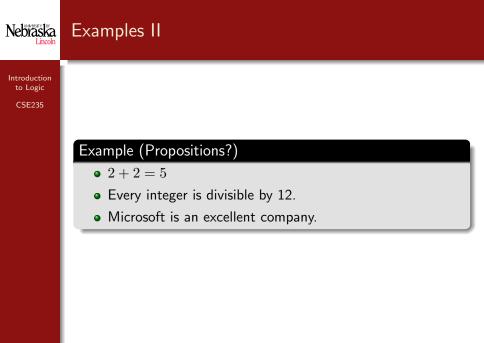
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Example (Propositions)

- Today is Monday.
- The derivative of $\sin x$ is $\cos x$.
- Every even number has at least two factors.

Example (Not Propositions)

- C++ is the best language.
- When is the pretest?
- Do your homework.



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Logical Connectives

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Connectives are used to create a *compound* proposition from two or more other propositions.

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- Negation (denoted ¬ or !)
- And (denoted \land) or Logical Conjunction
- Or (denoted \lor) or Logical Disjunction
- Exclusive Or (XOR, denoted \oplus)
- Implication (denoted \rightarrow)
- Biconditional; "if and only if" (denoted \leftrightarrow)



Negation

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A proposition can be negated. This is also a proposition. We usually denote the negation of a proposition p by $\neg p.$

Example (Negated Propositions)

- Today is *not* Monday.
- It is not the case that today is Monday.
- It is not the case that the derivative of $\sin x$ is $\cos x$.

Truth table:

$$\begin{array}{c|c}
p & \neg p \\
\hline
0 & 1 \\
1 & 0
\end{array}$$

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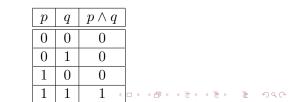


Logical And

Introduction to Logic CSE235 The logical connective A_{ND} is true only if *both* of the propositions are true. It is also referred to as a *conjunction*.

Example (Logical Connective: AND)

- It is raining and it is warm.
- $(2+3=5) \wedge (\sqrt{2} < 2)$
- Schrödinger's cat is dead and Schrödinger's cat is not dead.





Logical Or

Introduction to Logic CSE235 The logical disjunction (or logical or) is true if one or both of the propositions are true.

Example (Logical Connective: OR)

- It is raining or it is the second day of lecture.
- $(2+2=5) \lor (\sqrt{2} < 2)$
- You may have cake or ice cream.¹



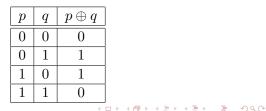


Exclusive Or

Introduction to Logic CSE235 The exclusive or of two propositions is true when exactly *one* of its propositions is true and the other one is false.

Example (Logical Connective: Exclusive Or)

- The circuit is either is on or off.
- Let ab < 0, then either a < 0 or b < 0 but not both.
- You may have cake or ice cream, but not both.





Implications I

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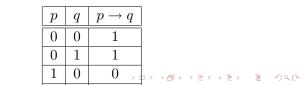
Definition

Let p and q be propositions. The implication

 $p \to q$

is the proposition that is false when \boldsymbol{p} is true and \boldsymbol{q} is false and true otherwise.

Here, p is called the "hypothesis" (or "antecedent" or "premise") and q is called the "conclusion" or "consequence".



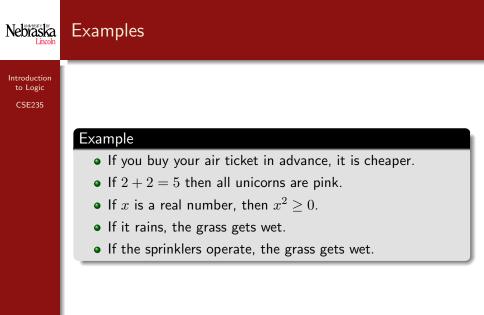


Implications II

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The implication $p \rightarrow q$ can be equivalently read as

- if p then q
- p implies q
- $\bullet\,$ if $p,\,q$
- p only if q
- q if p
- \bullet q when p
- \bullet q whenever p
- p is a sufficient condition for q (p is sufficient for q)
- q is a necessary condition for p (q is necessary for p)
- q follows from p



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Which of the following implications is true?

• If -1 is a positive number, then 2+2=5.

• If -1 is a positive number, then 2+2=4.

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• If $\sin x = 0$ then x = 0.



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Which of the following implications is true?

If -1 is a positive number, then 2 + 2 = 5.
 true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.

- If -1 is a positive number, then 2+2=4.
- If $\sin x = 0$ then x = 0.



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Which of the following implications is true?

If -1 is a positive number, then 2 + 2 = 5.
 true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.

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- If -1 is a positive number, then 2 + 2 = 4. true: for the same reason as above
- If $\sin x = 0$ then x = 0.



Introduction to Logic CSE235

Which of the following implications is true?

- If -1 is a positive number, then 2 + 2 = 5.
 true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.
- If −1 is a positive number, then 2 + 2 = 4.
 true: for the same reason as above
- If sin x = 0 then x = 0.
 false: x can be any multiple of π; i.e. if we let x = 2π then clearly sin x = 0, but x ≠ 0. The implication "if sin x = 0 then x = kπ for some integer k" is true.



Biconditional

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Definition

The biconditional

$$p \leftrightarrow q$$

is the proposition that is true when $p \mbox{ and } q$ have the same truth values. It is false otherwise.

Note that it is equivalent to $(p \rightarrow q) \land (q \rightarrow p)$

Truth table:

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

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Examples

Introduction to Logic CSE235

- $p \leftrightarrow q$ can be equivalently read as
 - p if and only if q
 - $\bullet \ p$ is necessary and sufficient for q
 - if p then q, and conversely
 - p iff q (Note typo in textbook, page 9, line 3.)

Example

- x > 0 if and only if x^2 is positive.
- The alarm goes off iff a burglar breaks in.
- $\bullet\,$ You may have pudding if and only if you eat your meat.^1

¹How can you have any pudding if you don't eat your meat? $= 9 \circ \circ$



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Which of the following biconditionals is true?

• $x^2 + y^2 = 0$ if and only if x = 0 and y = 0

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- 2+2=4 if and only if $\sqrt{2}<2$
- $x^2 \ge 0$ if and only if $x \ge 0$.



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Which of the following biconditionals is true?

• $x^2 + y^2 = 0$ if and only if x = 0 and y = 0true: both implications hold.

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• 2+2=4 if and only if $\sqrt{2}<2$

•
$$x^2 \ge 0$$
 if and only if $x \ge 0$.



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• 2 + 2 = 4 if and only if $\sqrt{2} < 2$ true: for the same reason above.

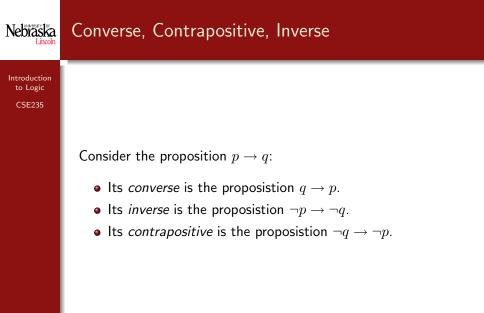
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$$x^2 \ge 0$$
 if and only if $x \ge 0$.



Introduction to Logic CSE235

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if x = 0 and y = 0true: both implications hold.
- 2 + 2 = 4 if and only if $\sqrt{2} < 2$ true: for the same reason above.
- $x^2 \ge 0$ if and only if $x \ge 0$. false: The converse holds. That is, "if $x \ge 0$ then $x^2 \ge 0$ ". However, the implication is false; consider x = -1. Then the hypothesis is true, $1^2 \ge 0$ but the conclusion fails.



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Truth Tables I

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Truth Tables are used to show the relationship between the truth values of individual propositions and the compound propositions based on them.

p	q	$p \wedge q$	$p \vee q$	$p\oplusq$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Table: Truth Table for Logical Conjunction, Disjunction, ExclusiveOr, and Implication



Introduction to Logic CSE235

Construct the Truth Table for the following compound proposition.

 $((p \wedge q) \vee \neg q)$

p	q	$p \wedge q$	$\neg q$	$((p \land q) \lor \neg q)$
0	0			
0	1			
1	0			
1	1			



Introduction to Logic CSE235

Construct the Truth Table for the following compound proposition.

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p	q	$p \wedge q$	$\neg q$	$((p \land q) \lor \neg q)$
0	0	0		
0	1	0		
1	0	0		
1	1	1		

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Construct the Truth Table for the following compound proposition.

 $((p \wedge q) \vee \neg q)$

p	q	$p \wedge q$	$\neg q$	$((p \land q) \lor \neg q)$
0	0	0	1	
0	1	0	0	
1	0	0	1	
1	1	1	0	



Introduction to Logic CSE235

Construct the Truth Table for the following compound proposition.

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0	1	0	0	0
1	0	0	1	1
1	1	1	0	1



Precedence of Logical Operators

Introduction to Logic CSE235 Just as in arithmetic, an ordering must be imposed on the use of logical operators in compound propositions.

Of course, parentheses can be used to make operators disambiguous:

$$\neg p \lor q \land \neg r \equiv (\neg p) \lor (q \land (\neg r))$$

But to avoid using unnecessary parentheses, we define the following precedences:

- $\bullet (\neg) \text{ Negation}$
- **2** (\land) Conjunction
- ${\small \textcircled{0}} \ (\lor) \ {\small \textsf{Disjunction}}$
- $\textcircled{O} (\rightarrow) \text{ Implication}$
- $(\leftrightarrow) \ \mathsf{Biconditional}$



Introduction to Logic CSE235

Logic is more precise than natural language:

- You may have cake or ice cream. Can I have both?
- If you buy your air ticket in advance, it is cheaper. Are there or not cheap last-minute tickets?

For this reason, logic is used for hardware and software *specification*.

Given a set of logic statements, one can decide whether or not they are satisfiable (i.e., consistent), although this is a costly process...



Bitwise Operations

Introduction to Logic CSE235 Computers represent information as bits (binary digits).

A *bit string* is a sequence of bits, the length of the string is the number of bits in the string.

Logical connectives can be applied to bit strings (of equal length). To do this, we simply apply the connective rules to each bit of the string:

Example

0110	1010	1101	
0101	0010	1111	
0111	1010	1111	bitwise OR
0100	0010	1101	bitwise AND
0011	1000	0010	bitwise XOR

A Boolean variable is a variable that can have value 0 for $1 = -9 \circ c$



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What is SAT? SAT is the problem of determining whether or not a sentence in propositional logic (PL) is satisfiable. Characterizing SAT as an NP-complete problem is at the foundation of Theoretical Computer Science.

Defining SAT

- Given: a PL sentence.
- Question: Determine whether it is satisfiable or not.

What is a PL sentence? What does satisfiable mean?



Logic in Theorerical Computer Science A sentence in PL

Introduction to Logic CSE235

- A sentence in PL is a *conjunction* of clauses
- A clause is a *disjunction* of literals
- A literal is a term or its negation
- A term is a (Boolean) variable (or proposition)

Example: $(a \lor b \lor \neg c \lor \neg d) \land (\neg b \lor c) \land (\neg a \lor c \lor d)$

A sentence in PL is a *satisfiable* iff we can assign truth value to the Boolean variables such that the sentence evaluates to true (i.e., holds).

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Logic in Programming Programming Example I

Introduction to Logic CSE235

Say you need to define a conditional statement as follows: "Increment x if all of the following conditions hold: $x>0, \ x<10$ and x=10."

You may try:

if(0<x<10 OR x=10) x++;

But is not valid in C++ or Java. How can you modify this statement by using a logical equivalence?

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Answer:



Logic in Programming Programming Example I

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Answer:

```
if(x>0 AND x<=10) x++;
```



Logic In Programming Programming Example II

Introduction to Logic CSE235 Say we have the following loop:

```
while
  ((i<size AND A[i]>10) OR
   (i<size AND A[i]<0) OR
   (i<size AND (NOT (A[i]!= 0 AND NOT (A[i]>= 10))
```

Is this good code? Keep in mind:

- Readability.
- Extraneous code is inefficient and poor style.
- Complicated code is more prone to errors and difficult to debug.



Propositional Equivalences Introduction

Introduction to Logic CSE235

To manipulate a set of statements (here, logical propositions) for the sake mathematical argumentation, an important step is to replace one statement with another equivalent statement (i.e., with the same truth value).

Below, we discuss:

- Terminology
- Establising logical equivalences using truth tables
- Establising logical equivalences using known laws (of logical equivalences)



Terminology Tautologies, Contradictions, Contingencies

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Definition

- A compound proposition that is always true, no matter what the truth values of the propositions that occur in it is called a *tautology*.
- A compound proposition that is always false is called a *contradiction*.
- Finally, a proposition that is neither a tautology nor a contradiction is called a *contingency*.

Example

A simple tautology is $p \lor \neg p$ A simple contradiction is $p \land \neg p$



Logical Equivalences Definition

Introduction to Logic CSE235

Definition

Propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

Informally, $p \ {\rm and} \ q$ are logically equivalent if whenever p is true, q is true, and vice versa.

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Notation $p \equiv q$ ("p is equivalent to q"), $p \iff q, p \Leftrightarrow q$, $p \Leftrightarrow q$.

Alort: \equiv is **not** a logical connective.



Introduction to Logic CSE235

Are and $p \rightarrow q$ and $\neg p \lor q$ logically equivalent?

To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \lor q$
0	0			
0	1			
1	0			
1	1			



Introduction to Logic CSE235

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1	1	1	0	



Introduction to Logic CSE235

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0	1	1	1	1
1	0	0	0	0
1	1	1	0	1



Introduction to Logic CSE235

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p	q	$p \rightarrow q$	$\neg p$	$\neg p \lor q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

The two columns in the truth table are identical, thus we conclude that

$$p \to q \equiv \neg p \lor q$$



Another Example

Introduction to Logic CSE235

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

p	q	r	$p \rightarrow r$	$(q \rightarrow r)$	$(p \to r) \lor (q \to r)$
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			



Another Example

Introduction to Logic CSE235

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p	q	r	$p \rightarrow r$	$(q \rightarrow r)$	$(p \to r) \lor (q \to r)$
0	0	0	1		
0	0	1	1		
0	1	0	1		
0	1	1	1		
1	0	0	0		
1	0	1	1		
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Introduction to Logic CSE235

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0	0	1	1	1	
0	1	0	1	0	
0	1	1	1	1	
1	0	0	0	1	
1	0	1	1	1	
1	1	0	0	0	
1	1	1	1	1	



Introduction to Logic CSE235

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

p	q	r	$p \rightarrow r$	$(q \rightarrow r)$	$(p \to r) \lor (q \to r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1



$\underset{\text{Continued}}{\text{Another Example}}$

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Now let's do it for $(p \land q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \land q) \to r$
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		



$\underset{\text{Continued}}{\text{Another Example}}$

Introduction to Logic CSE235

Now let's do it for $(p \land q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \land q) \to r$
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	1	
1	1	1	1	



$\underset{\text{Continued}}{\text{Another Example}}$

Introduction to Logic CSE235

Now let's do it for $(p \land q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \land q) \to r$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1



Another Example

Introduction to Logic CSE235

Now let's do it for $(p \land q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \land q) \to r$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

The truth values are identical, so we conclude that the logical equivalence holds.



Introduction to Logic CSE235

Tables of logical equivalences can be found in Rosen (page 24).

These and other can be found in a handout on the course web page http://www.cse.unl.edu/~cse235/files/ LogicalEquivalences.pdf

Let's take a quick look at this Cheat Sheet



Introduction to Logic CSE235

Logical equivalences can be used to construct additional logical equivalences.

Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$((p \wedge q) \rightarrow q) \iff \neg (p \wedge q) \lor q$$
 Implication Law



Introduction to Logic CSE235

Logical equivalences can be used to construct additional logical equivalences.

Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$\begin{array}{ccc} ((p \wedge q) \to q) & \Longleftrightarrow & \neg (p \wedge q) \lor q & \text{Implication Law} \\ & \Longleftrightarrow & (\neg p \lor \neg q) \lor q & \text{De Morgan's Law (1st)} \end{array}$$



Introduction to Logic CSE235

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Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$\begin{array}{lll} ((p \wedge q) \to q) & \Longleftrightarrow & \neg (p \wedge q) \lor q & \text{Implication Law} \\ & \Leftrightarrow & (\neg p \lor \neg q) \lor q & \text{De Morgan's Law (1st)} \\ & \Leftrightarrow & \neg p \lor (\neg q \lor q) & \text{Associative Law} \end{array}$$



Introduction to Logic CSE235

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Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$\begin{array}{lll} ((p \wedge q) \to q) & \Longleftrightarrow & \neg (p \wedge q) \lor q & \text{Implication Law} \\ & \Leftrightarrow & (\neg p \lor \neg q) \lor q & \text{De Morgan's Law (1st)} \\ & \Leftrightarrow & \neg p \lor (\neg q \lor q) & \text{Associative Law} \\ & \Leftrightarrow & \neg p \lor 1 & \text{Negation Law} \end{array}$$



Introduction to Logic CSE235

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Introduction to Logic CSE235 Example (Exercise $27)^1$: Show that

$$\neg (p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

¹This appears in Table 7, but you are not allowed to use it to prove it oqe



Introduction to Logic CSE235 Example (Exercise $27)^1$: Show that

$$\neg (p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

$$\iff \quad (p \to \neg q) \land (\neg q \to p) \qquad \quad \text{Equivalence Law}$$

¹This appears in Table 7, but you are not allowed to use it to prove it o q e



Introduction to Logic CSE235 Example (Exercise $27)^1$: Show that

$$\neg (p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

 $\begin{array}{ll} \Longleftrightarrow & (p \to \neg q) \land (\neg q \to p) \\ \Leftrightarrow & (\neg p \lor \neg q) \land (q \lor p) \end{array}$

Equivalence Law Implication Law

¹This appears in Table 7, but you are not allowed to use it to prove it 0 < 0



Introduction to Logic CSE235 Example (Exercise 27)¹: Show that

$$\neg (p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

 $\begin{array}{ll} \Longleftrightarrow & (p \to \neg q) \land (\neg q \to p) & \text{Equivalence Law} \\ \Leftrightarrow & (\neg p \lor \neg q) \land (q \lor p) & \text{Implication Law} \\ \Leftrightarrow & \neg (\neg ((\neg p \lor \neg q) \land (q \lor p))) & \text{Double Negation} \end{array}$

¹This appears in Table 7, but you are not allowed to use it to prove it 0 < 0



Introduction to Logic

Example (Exercise 27)¹: Show that

$$\neg (p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

 $\iff (p \to \neg q) \land (\neg q \to p)$ $\iff (\neg p \lor \neg q) \land (q \lor p)$ $\iff \neg (\neg ((\neg p \lor \neg q) \land (q \lor p)))$ Double Negation $\iff \neg (\neg (\neg p \lor \neg q) \lor \neg (q \lor p))$

Equivalence Law Implication Law De Morgan's Law

¹This appears in Table 7, but you are not allowed to use it to prove it $n < \infty$



Introduction to Logic CSE235 Example (Exercise $27)^1$: Show that

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 $\begin{array}{lll} \Longleftrightarrow & (p \to \neg q) \land (\neg q \to p) \\ \Leftrightarrow & (\neg p \lor \neg q) \land (q \lor p) \\ \Leftrightarrow & \neg (\neg ((\neg p \lor \neg q) \land (q \lor p))) \\ \Leftrightarrow & \neg (\neg (\neg p \lor \neg q) \lor \neg (q \lor p)) \\ \Leftrightarrow & \neg ((p \land q) \lor (\neg q \land \neg p)) \end{array}$

Equivalence Law Implication Law Double Negation De Morgan's Law De Morgan's Law

¹This appears in Table 7, but you are not allowed to use it to prove it 0 < 0



Introduction to Logic CSE235 Example (Exercise $27)^1$: Show that

$$\neg (p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

 $\begin{array}{lll} \Longleftrightarrow & (p \to \neg q) \land (\neg q \to p) \\ \Leftrightarrow & (\neg p \lor \neg q) \land (q \lor p) \\ \Leftrightarrow & \neg (\neg ((\neg p \lor \neg q) \land (q \lor p))) \\ \Leftrightarrow & \neg (\neg (\neg p \lor \neg q) \lor \neg (q \lor p)) \\ \Leftrightarrow & \neg ((p \land q) \lor (\neg q \land \neg p)) \\ \Leftrightarrow & \neg ((p \land q) \lor (\neg p \land \neg q)) \end{array}$

Equivalence Law Implication Law Double Negation De Morgan's Law De Morgan's Law Commutative Law

¹This appears in Table 7, but you are not allowed to use it to prove it 0 < 0



Introduction to Logic CSE235 Example (Exercise $27)^1$: Show that

$$\neg (p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

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Equivalence Law Implication Law Double Negation De Morgan's Law De Morgan's Law Commutative Law Equivalence Law (See Table 7, p24)

¹This appears in Table 7, but you are not allowed to use it to prove it $0 < \infty$



Introduction to Logic CSE235

Show that

$$\neg (q \to p) \lor (p \land q) \iff q$$

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 $\neg(q \to p) \lor (p \land q)$



Introduction to Logic CSE235

Show that

$$\neg(q \to p) \lor (p \land q) \iff q$$

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 $\neg(q \to p) \lor (p \land q)$

 $\iff \ \ (\neg (\neg q \lor p)) \lor (p \land q) \quad \text{Implication Law}$



Introduction to Logic CSE235

Show that

$$\neg(q \to p) \lor (p \land q) \iff q$$

 $\neg(q \to p) \lor (p \land q)$

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Introduction to Logic CSE235

Show that

$$\neg(q \to p) \lor (p \land q) \iff q$$

 $\neg(q \to p) \lor (p \land q)$

 $\begin{array}{ll} \Longleftrightarrow & (\neg(\neg q \lor p)) \lor (p \land q) & \text{Implication Law} \\ \Leftrightarrow & (q \land \neg p) \lor (p \land q) & \text{De Morgan's \& Double Negation} \\ \Leftrightarrow & (q \land \neg p) \lor (q \land p) & \text{Commutative Law} \end{array}$



Introduction to Logic CSE235

Show that

$$\neg(q \to p) \lor (p \land q) \iff q$$

 $\neg(q \to p) \lor (p \land q)$

$$\begin{array}{ll} \Longleftrightarrow & (\neg(\neg q \lor p)) \lor (p \land q) \\ \Leftrightarrow & (q \land \neg p) \lor (p \land q) \\ \Leftrightarrow & (q \land \neg p) \lor (q \land p) \\ \Leftrightarrow & q \land (\neg p \lor p) \end{array}$$

Implication Law De Morgan's & Double Negation Commutative Law Distributive Law

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Introduction to Logic CSE235

Show that

$$\neg(q \to p) \lor (p \land q) \iff q$$

 $\neg(q \to p) \lor (p \land q)$

$$\begin{array}{ll} \Longleftrightarrow & (\neg(\neg q \lor p)) \lor (p \land q) \\ \Leftrightarrow & (q \land \neg p) \lor (p \land q) \\ \Leftrightarrow & (q \land \neg p) \lor (q \land p) \\ \Leftrightarrow & q \land (\neg p \lor p) \\ \Leftrightarrow & q \land 1 \end{array}$$

Implication Law De Morgan's & Double Negation Commutative Law Distributive Law Cancelation Law

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Introduction to Logic CSE235

Show that

$$\neg(q \to p) \lor (p \land q) \iff q$$

 $\neg(q \to p) \lor (p \land q)$

$$\begin{array}{ll} \Longleftrightarrow & (\neg(\neg q \lor p)) \lor (p \land q) \\ \Leftrightarrow & (q \land \neg p) \lor (p \land q) \\ \Leftrightarrow & (q \land \neg p) \lor (q \land p) \\ \Leftrightarrow & q \land (\neg p \lor p) \\ \Leftrightarrow & q \land 1 \\ \Leftrightarrow & q \end{array}$$

Implication Law De Morgan's & Double Negation Commutative Law Distributive Law Cancelation Law Identity Law

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Logic In Programming Programming Example II Revisited

Introduction to Logic CSE235

Recall the loop:

while((i<size AND A[i]>10) OR
 (i<size AND A[i]<0) OR
 (i<size AND (NOT (A[i]!= 0 AND NOT (A[i]>=

Now, using logical equivalences, simplify it.



Logic In Programming Programming Example II Revisited

Introduction to Logic CSE235

Answer: Use De Morgan's Law and Distributivity.

Notice the ranges of all four conditions on A[i]; they can be merged and we can further simplify it to:



Programming Pitfall Note

Introduction to Logic CSE235

In C, C++ and Java, applying the commutative law is not such a good idea. These languages (compiler dependent) sometimes use "short-circuiting" for efficiency (at the machine level). For example, consider accessing an integer array A of size n.

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```
if(i<n && A[i]==0) i++;
```

is not equivalent to

```
if(A[i]==0 && i<n) i++;
```