Functions

Slides by Christopher M. Bourke
Instructor: Berthe Y. Choueiry

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cse235@cse.unl.edu

Introduction

You’ve already encountered functions throughout your education.

\[ f(x, y) = x + y \]
\[ f(x) = x \]
\[ f(x) = \sin x \]

Here, however, we will study functions on discrete domains and ranges. Moreover, we generalize functions to mappings. Thus, there may not always be a “nice” way of writing functions like above.

Definition

Function

A function \( f \) from a set \( A \) to a set \( B \) is an assignment of exactly one element of \( B \) to each element of \( A \). We write \( f(a) = b \) if \( b \) is the unique element of \( B \) assigned by the function \( f \) to the element \( a \in A \). If \( f \) is a function from \( A \) to \( B \), we write

\[ f : A \to B \]

This can be read as “\( f \) maps \( A \) to \( B \)”.

Note the subtlety:

- Each and every element in \( A \) has a single mapping.
- Each element in \( B \) may be mapped to by several elements in \( A \) or not at all.

Definitions

Terminology

Definition

Let \( f : A \to B \) and let \( f(a) = b \). Then we use the following terminology:

- \( A \) is the domain of \( f \), denoted \( \text{dom}(f) \).
- \( B \) is the codomain of \( f \).
- \( b \) is the image of \( a \).
- \( a \) is the preimage of \( b \).
- The range of \( f \) is the set of all images of elements of \( A \), denoted \( \text{rng}(f) \).

Definitions

Visualization

A function, \( f : A \to B \).

A function, \( f : A \to B \).

Definition I

More Definitions

Definition

Let \( f_1 \) and \( f_2 \) be functions from a set \( A \) to \( \mathbb{R} \). Then \( f_1 + f_2 \) and \( f_1f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by

\[ (f_1 + f_2)(x) = f_1(x) + f_2(x) \]
\[ (f_1f_2)(x) = f_1(x)f_2(x) \]

Example
**Definition II**

Let \( f_1(x) = x^4 + 2x^2 + 1 \) and \( f_2(x) = 2 - x^2 \) then
\[
(f_1 + f_2)(x) = (x^4 + 2x^2 + 1) + (2 - x^2) = x^4 + 2x^2 + 3
\]
\[
(f_1f_2)(x) = (x^4 + 2x^2 + 1) \cdot (2 - x^2) = -x^6 + 3x^4 + 2
\]

**Definition III**

Let \( f : A \to B \) and let \( S \subseteq A \). The **image of** \( S \) is the subset of \( B \) that consists of all the images of the elements of \( S \). We denote the image of \( S \) by \( f(S) \), so that
\[
f(S) = \{ f(s) \mid s \in S \}
\]

Note that here, an image is a set rather than an element.

**Example**

Let

- \( A = \{a_1, a_2, a_3, a_4, a_5\} \)
- \( B = \{b_1, b_2, b_3, b_4\} \)
- \( f = \{(a_1, b_1), (a_2, b_3), (a_3, b_1), (a_4, b_1), (a_5, b_4)\} \)
- \( S = \{a_1, a_3\} \)

Draw a diagram for \( f \).

The image of \( S \) is \( f(S) = \{b_2, b_3\} \)

**Definition IV**

A function \( f \) whose domain and codomain are subsets of the set of real numbers is called **strictly increasing** if \( f(x) < f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \). A function \( f \) is called **strictly decreasing** if \( f(x) > f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \).

### Injections, Surjections, Bijections I

**Definition**

A function \( f \) is said to be one-to-one (or **injective**) if
\[
f(x) = f(y) \Rightarrow x = y
\]
for all \( x \) and \( y \) in the domain of \( f \). A function is an **injection** if it is one-to-one.

Intuitively, an injection simply means that each element in \( A \) uniquely maps to an element in \( B \).

It may be useful to think of the contrapositive of this definition:
\[
x \neq y \Rightarrow f(x) \neq f(y)
\]

### Injections, Surjections, Bijections II

**Definition**

A function \( f : A \to B \) is called **onto** (or **surjective**) if for every element \( b \in B \) there is an element \( a \in A \) with \( f(a) = b \). A function is called a **surjection** if it is onto.

Again, intuitively, a surjection means that every element in the codomain is mapped. This implies that the range is the same as the codomain.

### Injections, Surjections, Bijections III

**Definition**

A function \( f \) is a **one-to-one correspondence** (or a **bijection** if it is both one-to-one and onto.

One-to-one correspondences are important because they endow a function with an **inverse**. They also allow us to have a concept of cardinality for infinite sets!

Let’s take a look at a few general examples to get the feel for these definitions.
Function Examples
A Non-function

This is not a function: Both $a_1$ and $a_2$ map to more than one element in $B$.

Function Examples
A Function; Neither One-To-One Nor Onto

This function not one-to-one since $a_1$ and $a_3$ both map to $b_1$. It is not onto either since $b_4$ is not mapped to by any element in $A$.

Function Examples
One-To-One, Not Onto

This function is one-to-one since every $a_i \in A$ maps to a unique element in $B$. However, it is not onto since $b_4$ is not mapped to by any element in $A$.

Function Examples
Onto, Not One-To-One

This function is onto since every element $b_i \in B$ is mapped to by some element in $A$. However, it is not one-to-one since $b_3$ is mapped to more than one element in $A$.

Function Examples
A Bijection

This function is a bijection because it is both one-to-one and onto; every element in $A$ maps to a unique element in $B$ and every element in $B$ is mapped by some element in $A$.

Exercises I
Exercise I

Example

Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = 2x - 3$$

What is the domain and range of $f$? Is it onto? One-to-one?

Clearly, $\text{dom}(f) = \mathbb{Z}$. To see what the range is, note that

\[ b \in \text{rng}(f) \iff b = 2(a - 2) + 1 \iff b \text{ is odd} \]

\[ a \in \mathbb{Z} \]

\[ \iff b = 2a - 3 \]
Exercises II
Exercise I

Therefore, the range is the set of all odd integers. Since the range and codomain are different, we can also conclude that \( f \) is not onto.

However, \( f \) is one-to-one. To prove this, note that
\[
f(x_1) = f(x_2) \Rightarrow 2x_1 - 3 = 2x_2 - 3
\Rightarrow x_1 = x_2
\]
follows from simple algebra.

Exercises I
Exercise III

Example
Define \( f : \mathbb{Z} \to \mathbb{Z} \) by
\[
f(x) = x^2 - 5x + 5
\]
Is this function one-to-one? Onto?

It is not one-to-one since for
\[
f(x_1) = f(x_2) \Rightarrow x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2
\Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2
\Rightarrow (x_1 + x_2)(x_1 - x_2) = 5(x_1 - x_2)
\Rightarrow x_1 + x_2 = 5
\]

Exercises II
Exercise IV

Example
Define \( f : \mathbb{Z} \to \mathbb{Z} \) by
\[
f(x) = 2x^2 + 7x
\]
Is this function one-to-one? Onto?

Again, since this is a parabola, it cannot be onto (where is the global minimum?).
**Inverse Functions I**

**Definition**

Let \( f : A \to B \) be a bijection. The inverse function of \( f \) is the function that assigns to an element \( b \in B \) the unique element \( a \in A \) such that \( f(a) = b \). The inverse function of \( f \) is denoted by \( f^{-1} \). Thus \( f^{-1}(b) = a \) when \( f(a) = b \).

More succinctly, if an inverse exists, 
\[
 f(a) = b \iff f^{-1}(b) = a
\]

**Examples**

**Example I**

Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by
\[
 f(x) = 2x - 3
\]

What is \( f^{-1} \)?

First, verify that \( f \) is a bijection (it is). To find an inverse, we use substitution:

- Let \( f^{-1}(y) = x \)
- Let \( y = 2x - 3 \) and solve for \( x \)
- Clearly, \( x = \frac{y+3}{2} \) so,
  \[
  f^{-1}(y) = \frac{y+3}{2}.
  \]
Examples

Example II

**Example**

Let

\[ f(x) = x^2 \]

What is \( f^{-1} \)?

No domain/codomain has been specified. Say \( f : \mathbb{R} \to \mathbb{R} \) is \( f \) a bijection? Does an inverse exist?

No, however if we specify that

\[ A = \{ x \in \mathbb{R} \mid x \leq 0 \} \]

and

\[ B = \{ y \in \mathbb{R} \mid y \geq 0 \} \]

then it becomes a bijection and thus has an inverse.

Examples

Example III

**Example**

Let

\[ f(x) = 2^x \]

What should the domain/codomain be for this to be a bijection?

What is the inverse?

The function should be \( f : \mathbb{R} \to \mathbb{R}^+ \). What happens when we include 0? Restrict either one to \( \mathbb{Z} \)?

Let \( f^{-1}(y) = x \) and \( y = 2^x \), solving for \( x \) we get \( x = \log_2(y) \).

Therefore,

\[ f^{-1}(y) = \log_2(y) \]

Composition II

Note the order that you apply a function matters—you go from inner most to outer most.

The composition \( f \circ g \) cannot be defined unless the the range of \( g \) is a subset of the domain of \( f \);

\[ f \circ g \text{ is defined } \iff \text{rng}(g) \subseteq \text{dom}(f) \]

It also follows that \( f \circ g \) is not necessarily the same as \( g \circ f \).

Composition of Functions

Figure

\[ (f \circ g)(a) \]

The composition of two functions.
Composition
Example I

Let \( f \) and \( g \) be functions, \( \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
\begin{align*}
  f(x) &= 2x - 3 \\
  g(x) &= x^2 + 1
\end{align*}
\]
What are \( f \circ g \) and \( g \circ f \)?

Note that \( f \) is bijective, thus \( \text{dom}(f) = \text{rng}(f) = \mathbb{R} \). For \( g \), we have that \( \text{dom}(g) = \mathbb{R} \) but that \( \text{rng}(g) = \{ x \in \mathbb{R} \mid x \geq 1 \} \).

Equality

Though intuitive, we formally state what it means for two functions to be equal.

Lemma
Two functions \( f \) and \( g \) are equal if and only if \( \text{dom}(f) = \text{dom}(g) \) and
\[
\forall a \in \text{dom}(f) (f(a) = g(a))
\]

Composition
Example I

Even so, \( \text{rng}(g) \subseteq \text{dom}(f) \) and so \( f \circ g \) is defined. Also, \( \text{rng}(f) \subseteq \text{dom}(g) \) so \( g \circ f \) is defined as well.

\[
\begin{align*}
  (f \circ g)(x) &= g(f(x)) \\
  &= g(2x - 3) \\
  &= (2x - 3)^2 + 1 \\
  &= 4x^2 - 12x + 10
\end{align*}
\]
and
\[
\begin{align*}
  (g \circ f)(x) &= f(g(x)) \\
  &= f(x^2 + 1) \\
  &= 2(x^2 + 1) - 3 \\
  &= 2x^2 - 1
\end{align*}
\]

Associativity

Though the composition of functions is not commutative \( (f \circ g \neq g \circ f) \), it is associative.

Lemma
Composition of functions is an associative operation; that is,
\[
(f \circ g) \circ h = f \circ (g \circ h)
\]

Important Functions

Identity Function

Definition
The identity function on a set \( A \) is the function
\[
i : A \rightarrow A
\]
defined by \( i(a) = a \) for all \( a \in A \). This symbol is the Greek letter iota.

One can view the identity function as a composition of a function and its inverse;
\[
i(a) = (f \circ f^{-1})(a)
\]
Moreover, the composition of any function \( f \) with the identity function is itself \( f \);
\[
(f \circ i)(a) = (i \circ f)(a) = f(a)
\]

Inverses & Identity

The identity function, along with the composition operation gives us another characterization for when a function has an inverse.

Theorem
Functions \( f : A \rightarrow B \) and \( g : B \rightarrow A \) are inverses if and only if
\[
g \circ f = i_A \text{ and } f \circ g = i_B
\]
That is,
\[
\forall a \in A, b \in B (g(f(a)) = a \wedge f(g(b)) = b)
\]
Important Functions I

Absolute Value Function

Definition

The absolute value function, denoted $|x|$ is a function $f : \mathbb{R} \to \{y \in \mathbb{R} \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Floor & Ceiling Functions

Definition

The floor function, denoted $\lfloor x \rfloor$ is a function $\mathbb{R} \to \mathbb{Z}$. Its value is the largest integer that is less than or equal to $x$.

The ceiling function, denoted $\lceil x \rceil$ is a function $\mathbb{R} \to \mathbb{Z}$. Its value is the smallest integer that is greater than or equal to $x$.

Graphical View

Factorial Function

Definition

The factorial function, denoted $n!$ is a function $\mathbb{N} \to \mathbb{Z}^+$. Its value is the product of the first $n$ positive integers.

$$n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

Stirling’s Approximation

The factorial function is defined on a discrete domain. In many applications, it is useful to consider a continuous version of the function (say if we want to differentiate it).

To this end, we have Stirling’s Formula:

$$n! \approx \sqrt{2 \pi n} \frac{n^n}{e^n}$$