Combinatorics

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Notes

Combinatorics I

Introduction

Combinatorics is the study of collections of objects. Specifically, *counting* objects, arrangement, derangement, etc. of objects along with their mathematical properties.

Counting objects is important in order to analyze algorithms and compute discrete probabilities.

Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability.

Notes

Combinatorics II

Introduction

A simple example: How many arrangements are there of a deck of 52 cards?

In addition, combinatorics can be used as a proof technique.

A *combinatorial proof* is a proof method that uses counting arguments to prove a statement.

Notes

Product Rule

If two events are not mutually exclusive (that is, we do them separately), then we apply the product rule.

Theorem (Product Rule)

Suppose a procedure can be accomplished with two disjoint subtasks. If there are n_1 ways of doing the first task and n_2 ways of doing the second, then there are

$$n_1 \cdot n_2$$

ways of doing the overall procedure.

Notes

Sum Rule I

If two events *are* mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule.

Theorem (Sum Rule)

If an event e_1 can be done in n_1 ways and an event e_2 can be done in n_2 ways and e_1 and e_2 are mutually exclusive, then the number of ways of both events occurring is

$$n_1 + n_2$$

Notes

Sum Rule II

There is a natural generalization to any sequence of m tasks; namely the number of ways m mutually exclusive events can occur is

$$n_1 + n_2 + \cdots n_{m-1} + n_m$$

We can give another formulation in terms of sets. Let $A_1, A_2, \ldots, A_m$ be pairwise *disjoint* sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

In fact, this is a special case of the general *Principle of Inclusion-Exclusion*.

Notes

Principle of Inclusion-Exclusion (PIE) I

Introduction

Say there are two events, e_1 and e_2 for which there are n_1 and n_2 possible outcomes respectively.

Now, say that only *one* event can occur, not both.

In this situation, we cannot apply the sum rule? Why?

Notes

Principle of Inclusion-Exclusion (PIE) II

Introduction

We cannot use the sum rule because we would be *over counting* the number of possible outcomes.

Instead, we have to count the number of possible outcomes of e_1 and e_2 *minus* the number of possible outcomes in common to both; i.e. the number of ways to do both "tasks".

If again we think of them as sets, we have

$$|A_1| + |A_2| - |A_1 \cap A_2|$$

Notes

Principle of Inclusion-Exclusion (PIE) III

Introduction

More generally, we have the following.

Lemma

Let A, B be subsets of a finite set U. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cap B| \leq \min\{|A|, |B|\}$
3. $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$
4. $|\overline{A}| = |U| - |A|$
5. $|A \oplus B| = |A \cup B| - |A \cap B| = A + B - 2|A \cap B| = |A \setminus B| + |B \setminus A|$
6. $|A \times B| = |A| \times |B|$

Notes

Principle of Inclusion-Exclusion (PIE) I

Theorem

Theorem

Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_i |A_i| \\ &- \sum_{i<j} |A_i \cap A_j| \\ &+ \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\ &- \cdots \\ &+ (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|\end{aligned}$$

Notes

Principle of Inclusion-Exclusion (PIE) II

Theorem

Each summation is over all i, pairs i, j with $i < j$, triples i, j, k with $i < j < k$ etc.

Notes

Principle of Inclusion-Exclusion (PIE) III

Theorem

To illustrate, when $n = 3$, we have

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &- \big[|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\big] \\ &+ |A_1 \cap A_2 \cap A_3|\end{aligned}$$

Notes

Principle of Inclusion-Exclusion (PIE) IV

Theorem

To illustrate, when $n = 4$, we have

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &- \Big[|A_1 \cap A_2| + |A_1 \cap A_3| + +|A_1 \cap A_4| \\ &\quad |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|\Big] \\ &+ \Big[|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \\ &\quad |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|\Big] \\ &- |A_1 \cap A_2 \cap A_3 \cap A_4|\end{aligned}$$

Notes

Principle of Inclusion-Exclusion (PIE) I

Example I

Example

How many integers between 1 and 300 (inclusive) are

1. Divisible by at least one of 3, 5, 7?
2. Divisible by 3 and by 5 but not by 7?
3. Divisible by 5 but by neither 3 nor 7?

Let

$$\begin{aligned}A &= \{n \mid 1 \leq n \leq 300 \wedge 3 \mid n\} \\ B &= \{n \mid 1 \leq n \leq 300 \wedge 5 \mid n\} \\ C &= \{n \mid 1 \leq n \leq 300 \wedge 7 \mid n\}\end{aligned}$$

Notes

Principle of Inclusion-Exclusion (PIE) II

Example I

How big are each of these sets? We can easily use the floor function;

$$\begin{aligned}|A| &= \lfloor 300/3 \rfloor = 100 \\ |B| &= \lfloor 300/5 \rfloor = 60 \\ |C| &= \lfloor 300/7 \rfloor = 42\end{aligned}$$

For (1) above, we are asked to find $|A \cup B \cup C|$.

Notes

Principle of Inclusion-Exclusion (PIE) III

Example I

By the principle of inclusion-exclusion, we have that

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -\Big[|A \cap B| + |A \cap C| + |B \cap C|\Big] \\ &\quad +|A \cap B \cap C| \end{aligned}$$

It remains to find the final 4 cardinalities.

All three divisors, 3, 5, 7 are relatively prime. Thus, any integer that is divisible by *both* 3 and 5 must simply be divisible by 15.

Notes

Principle of Inclusion-Exclusion (PIE) IV

Example I

Using the same reasoning for all pairs (and the triple) we have

$$\begin{aligned} |A \cap B| &= \lfloor 300/15 \rfloor = 20 \\ |A \cap C| &= \lfloor 300/21 \rfloor = 14 \\ |B \cap C| &= \lfloor 300/35 \rfloor = 8 \\ |A \cap B \cap C| &= \lfloor 300/105 \rfloor = 2 \end{aligned}$$

Therefore,

$$|A \cup B \cup C| = 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162$$

Notes

Principle of Inclusion-Exclusion (PIE) V

Example I

For (2) above, it is enough to find

$$|(A \cap B) \setminus C|$$

By the definition of set-minus,

$$|(A \cap B) \setminus C| = |A \cap B| - |A \cap B \cap C| = 20 - 2 = 18$$

Notes

Principle of Inclusion-Exclusion (PIE) VI

Example I

For (3) above, we are asked to find

$$|B \setminus (A \cup C)| = |B| - |B \cap (A \cup C)|$$

By distributing B over the intersection, we get

$$\begin{aligned} |B \cap (A \cup C)| &= |(B \cap A) \cup (B \cap C)| \\ &= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)| \\ &= |B \cap A| + |B \cap C| - |B \cap A \cap C| \\ &= 20 + 8 - 2 = 26 \end{aligned}$$

So the answer is $|B| - 26 = 60 - 26 = 34$.

Notes

Principle of Inclusion-Exclusion (PIE) I

Example II

The principle of inclusion-exclusion can be used to count the number of onto functions.

Theorem

Let A, B be non-empty sets of cardinality m, n with $m \geq n$. Then there are

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m$$

i.e. $\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^m$ onto functions $f : A \to B$.

See textbook page 460.

Notes

Principle of Inclusion-Exclusion (PIE) II

Example II

Example

How many ways of giving out 6 pieces of candy to 3 children if each child must receive at least one piece?

This can be modeled by letting A represent the set of candies and B be the set of children.

Then a function $f : A \to B$ can be interpreted as giving candy a_i to child c_j.

Since each child must receive at least one candy, we are considering only onto functions.

Notes

Principle of Inclusion-Exclusion (PIE) III

Example II

To count how many there are, we apply the theorem and get (for $m = 6, n = 3$),

$$3^6 - \binom{3}{1}(3-1)^6 + \binom{3}{2}(3-2)^6 = 540$$

Notes

Derangements I

Consider the hatcheck problem.

- An employee checks hats from n customers.
- However, he forgets to tag them.
- When customer's check-out their hats, they are given one at random.

What is the probability that no one will get their hat back?

Notes

Derangements II

This can be modeled using *derangements*: permutations of objects such that no element is in its original position.

For example, 21453 is a derangement of 12345, but 21543 is not.

Theorem

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{n!}\right]$$

See textbook page 461.

Notes

Derangements III

Thus, the answer to the hatcheck problem is

$$\frac{D_n}{n!}$$

Its interesting to note that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \cdots$$

So that the probability of the hatcheck problem converges;

$$\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1} = .368\ldots$$

Notes

The Pigeonhole Principle I

The *pigeonhole principle* states that if there are more pigeons than there are roosts (pigeonholes), for at least one pigeonhole, more than two pigeons must be in it.

Theorem (Pigeonhole Principle)

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two ore more objects.

This is a fundamental tool of elementary discrete mathematics. It is also known as the *Dirichlet Drawer Principle*.

Notes

The Pigeonhole Principle II

It is *seemingly* simple, but *very* powerful.

The difficulty comes in where and how to apply it.

Some simple applications in computer science:

- Calculating the probability of Hash functions having a collision.
- Proving that there can be *no* lossless compression algorithm compressing all files to within a certain ratio.

Lemma

For two finite sets A, B there exists a bijection $f : A \to B$ if and only if $|A| = |B|$.

Notes

Generalized Pigeonhole Principle I

Theorem

If N objects are placed into k boxes then there is at least one box containing at least

$$\left\lceil \frac{N}{k} \right\rceil$$

Example

In any group of 367 or more people, at least two of them must have been born on the same date.

Notes

Generalized Pigeonhole Principle II

A probabilistic generalization states that if n objects are randomly put into m boxes with uniform probability (each object is placed in a given box with probability $1/m$) then at least one box will hold more than one object with probability,

$$1 - \frac{m!}{(m-n)!m^n}$$

Notes

Generalized Pigeonhole Principle III

Example

Among 10 people, what is the probability that two or more will have the same birthday?

Here, $n = 10$ and $m = 365$ (ignore leapyears). Thus, the probability that two will have the same birthday is

$$1 - \frac{365!}{(365-10)!365^{10}} \approx .1169$$

So less than a 12% probability!

Notes

Pigeonhole Principle I

Example I

Example

Show that in a room of n people with certain acquaintances, some pair must have the same number of acquaintances.

Note that this is equivalent to showing that any symmetric, irreflexive relation on n elements must have two elements with the same number of relations.

We'll show by contradiction using the pigeonhole principle.

Assume to the contrary that every person has a different number of acquaintances; $0, 1, \ldots, n-1$ (we cannot have n here because it is irreflexive). Are we done?

Notes

Pigeonhole Principle II

Example I

No, since we only have n people, this is okay (i.e. there are n possibilities).

We need to use the fact that acquaintanceship is a symmetric, irreflexive relation.

In particular, some person knows 0 people while another knows $n-1$ people.

In other words, someone knows everyone, but there is also a person that knows no one.

Thus, we have reached a contradiction. $\square$

Notes

Pigeonhole Principle I

Example II

Example

Show that in any list of ten nonnegative integers, $A_0, \ldots, A_9$, there is a string of consecutive items of the list $a_l, a_{l+1}, \ldots$ whose sum is divisible by 10.

Consider the following 10 numbers.

$$\begin{aligned}
&a_0 \\
&a_0 + a_1 \\
&a_0 + a_1 + a_2 \\
&\vdots \\
&a_0 + a_1 + a_2 + \ldots + a_9
\end{aligned}$$

If any one of them is divisible by 10 then we are done.

Notes

Pigeonhole Principle II

Example II

Otherwise, we observe that each of these numbers must be in one of the congruence classes

$$1 \textbf{ mod } 10, 2 \textbf{ mod } 10, \ldots, 9 \textbf{ mod } 10$$

By the pigeonhole principle, at least two of the integers above must lie in the same congruence class. Say a, a' lie in the congruence class k **mod** 10.

Then

$$(a - a') \equiv k - k (\text{mod } 10)$$

and so the difference $(a - a')$ is divisible by 10. $\square$

Notes

Pigeonhole Principle I

Example III

Example

Say 30 buses are to transport 2000 Cornhusker fans to Colorado. Each bus has 80 seats. Show that

1. One of the buses will have 14 empty seats.
2. One of the buses will carry at least 67 passengers.

For (1), the total number of seats is $30 \cdot 80 = 2400$ seats. Thus there will be $2400 - 2000 = 400$ empty seats total.

Notes

Pigeonhole Principle II

Example III

By the generalized pigeonhole principle, with 400 empty seats among 30 buses, one bus will have at least

$$\left\lceil \frac{400}{30} \right\rceil = 14$$

empty seats.

For (2) above, by the pigeonhole principle, seating 2000 passengers among 30 buses, one will have at least

$$\left\lceil \frac{2000}{30} \right\rceil = 67$$

passengers.

Notes

Permutations I

A *permutation* of a set of distinct objects is an *ordered* arrangement of these objects. An ordered arrangement of r elements of a set is called an r-*permutation*.

Theorem

The number of r permutations of a set with n distinct elements is

$$P(n,r) = \prod_{i=0}^{r-1}(n-i) = n(n-1)(n-2)\cdots(n-r+1)$$

Notes

Permutations II

It follows that

$$P(n,r) = \frac{n!}{(n-r)!}$$

In particular,

$$P(n,n) = n!$$

Again, note here that *order is important*. It is necessary to distinguish in what cases order is important and in which it is not.

Notes

Permutations

Example I

Example

How many pairs of dance partners can be selected from a group of 12 women and 20 men?

The first woman can be partnered with any of the 20 men. The second with any of the remaining 19, etc.

To partner all 12 women, we have

$$P(20,12)$$

Notes

Permutations

Example II

Example

In how many ways can the English letters be arranged so that there are exactly ten letters between a and z?

The number of ways of arranging 10 letters between a and z is $P(24, 10)$. Since we can choose either a or z to come first, there are $2P(24, 10)$ arrangements of this 12-letter block.

For the remaining 14 letters, there are $P(15, 15) = 15!$ arrangements. In all, there are

$$2P(24, 10) \cdot 15!$$

Notes

Permutations

Example III

Example

How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern bge nor eaf?

The number of total permutations is $P(7, 7) = 7!$.

If we fix the pattern bge, then we can consider it as a single block. Thus, the number of permutations with this pattern is $P(5, 5) = 5!$.

Notes

Permutations

Example III - Continued

Fixing the pattern eaf we have the same number, $5!$.

Thus we have

$$7! - 2(5!)$$

Is this correct?

No. We have taken away too many permutations: ones containing *both* eaf and bge.

Here there are two cases, when eaf comes first and when bge comes first.

Notes

Permutations

Example III - Continued

eaf cannot come before bge, so this is not a problem.

If bge comes first, it must be the case that we have $bgeaf$ as a single block and so we have 3 blocks or $3!$ arrangements.

Altogether we have

$$7! - 2(5!) + 3! = 4806$$

Notes

Combinations I

Definition

Whereas permutations consider order, *combinations* are used when *order does not matter*.

Definition

An k-combination of elements of a set is an unordered selection of k elements from the set. A combination is simply a subset of cardinality k.

Notes

Combinations II

Definition

Theorem

The number of k-combinations of a set with cardinality n with $0 \leq k \leq n$ is

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Note: the notation, $\binom{n}{k}$ is read, "n choose k". In TEX use {`n choose k`} (with the forward slash).

Notes

Combinations III

Definition

A useful fact about combinations is that they are symmetric.

$$\binom{n}{1} = \binom{n}{n-1}$$

$$\binom{n}{2} = \binom{n}{n-2}$$

etc.

Notes

Combinations IV

Definition

This is formalized in the following corollary.

Corollary

Let n, k be nonnegative integers with $k \leq n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$

Notes

Combinations I

Example I

Example

In the Powerball lottery, you pick five numbers between 1 and 55 and a single "powerball" number between 1 and 42. How many possible plays are there?

Order here doesn't matter, so the number of ways of choosing five regular numbers is

$$\binom{55}{5}$$

Notes

Combinations II

Example I

We can choose among 42 power ball numbers. These events are not mutually exclusive, thus we use the product rule.

$$42\binom{55}{5} = 42\frac{55!}{(55-5)!5!} = 146,107,962$$

So the odds of winning are

$$\frac{1}{146,107,962} < .000000006845$$

Notes

Combinations I

Example II

Example

In a sequence of 10 coin tosses, how many ways can 3 heads and 7 tails come up?

The number of ways of choosing 3 heads out of 10 coin tosses is

$$\binom{10}{3}$$

Notes

Combinations II

Example II

However, this is the same as choosing 7 tails out of 10 coin tosses;

$$\binom{10}{3} = \binom{10}{7} = 120$$

This is a perfect illustration of the previous corollary.

Notes

Combinations I

Example III

Example

How many possible committees of five people can be chosen from 20 men and 12 women if

1. if exactly three men must be on each committee?
2. if at least four women must be on each committee?

Notes

Combinations II

Example III

For (1), we must choose 3 men from 20 then two women from 12. These are not mutually exclusive, thus the product rule applies.

$$\binom{20}{3}\binom{12}{2}$$

Notes

Combinations III

Example III

For (2), we consider two cases; the case where four women are chosen and the case where five women are chosen. These two cases *are* mutually exclusive so we use the addition rule.

For the first case we have

$$\binom{20}{1}\binom{12}{4}$$

Notes

Combinations IV

Example III

And for the second we have

$$\binom{20}{0}\binom{12}{5}$$

Together we have

$$\binom{20}{1}\binom{12}{4} + \binom{20}{0}\binom{12}{5} = 10,692$$

Notes

Binomial Coefficients I

Introduction

The number of r-combinations, $\binom{n}{r}$ is also called a *binomial coefficient*.

They are the coefficients in the expansion of the expression (multivariate polynomial), $(x+y)^n$. A *binomial* is a sum of two terms.

Notes