Asymptotics

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Introduction I

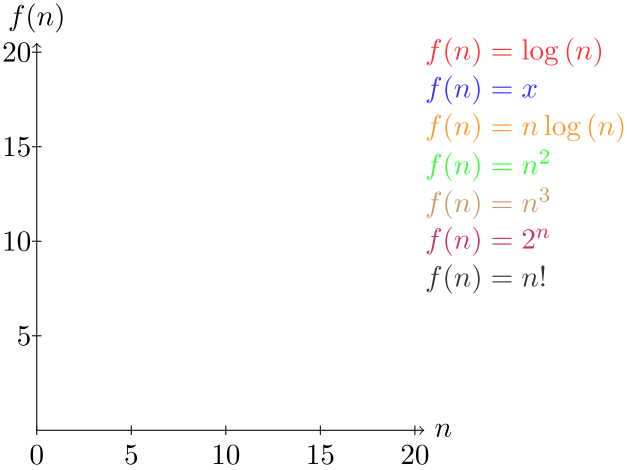
Recall that we are really only interested in the *Order of Growth* of an algorithm's complexity.

How well does the algorithm perform as the input size grows;

$$n \to \infty$$

We have seen how to mathematically evaluate the cost functions of algorithms with respect to their input size n and their elementary operation.

However, it suffices to simply measure a cost function's *asymptotic* behavior.


$f(n)$
20
15
10
5
0
5
10
15
20
n
$f(n) = \log(n)$
$f(n) = x$
$f(n) = n \log(n)$
$f(n) = n^2$
$f(n) = n^3$
$f(n) = 2^n$
$f(n) = n!$

Introduction I

In practice, specific hardware, implementation, languages, etc. will greatly affect how the algorithm behaves. However, we want to study and analyze algorithms *in and of themselves*, independent of such factors.

For example, an algorithm that executes its elementary operation $10n$ times is better than one which executes it $.005n^2$ times. Moreover, algorithms that have running times n^2 and $2000n^2$ are considered to be *asymptotically equivalent*.

Big-O Definition

Definition

Let f and g be two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$. We say that

$$f(n) \in \mathcal{O}(g(n))$$

(read: f is Big-"O" of g) if there exists a constant $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$,

$$f(n) \leq cg(n)$$

- Big-O is actually Omicron, but it suffices to write "O"
- Intuition: f is (*asymptotically*) less than or equal to g
- Big-O gives an asymptotic *upper bound*

Big-Omega Definition

Definition

Let f and g be two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$. We say that

$$f(n) \in \Omega(g(n))$$

(read: f is Big-Omega of g) if there exist $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$,

$$f(n) \geq cg(n)$$

- Intuition: f is (*asymptotically*) greater than or equal to g.
- Big-Omega gives an asymptotic *lower bound*.

Big-Theta Definition

Definition

Let f and g be two functions $f, g : \mathbb{N} \to \mathbb{R}^+$. We say that

$$f(n) \in \Theta(g(n))$$

(read: f is Big-Theta of g) if there exist constants $c_1, c_2 \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$,

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

- Intuition: f is (*asymptotically*) equal to g.
- f is bounded above *and* below by g.
- Big-Theta gives an asymptotic *equivalence*.

Asymptotic Properties I

Theorem

For $f_1(n) \in \mathcal{O}(g_1(n))$ and $f_2 \in \mathcal{O}(g_2(n))$,

$$f_1(n) + f_2(n) \in \mathcal{O}(\max\{g_1(n), g_2(n)\})$$

This property implies that we can ignore lower order terms. In particular, for any polynomial $p(n)$ with degree k, $p(n) \in \mathcal{O}(n^k)$.[1]

In addition, this gives us justification for ignoring constant coefficients. That is, for any function $f(n)$ and positive constant c,

$$cf(n) \in \Theta(f(n))$$

Asymptotic Properties II

Some obvious properties also follow from the definition.

Corollary

For positive functions, $f(n)$ and $g(n)$ the following hold:

- $f(n) \in \Theta(g(n)) \iff f(n) \in \mathcal{O}(g(n))$ *and* $f(n) \in \Omega(g(n))$
- $f(n) \in \mathcal{O}(g(n)) \iff g(n) \in \Omega(f(n))$

The proof is left as an exercise.

[1] More accurately, $p(n) \in \Theta(n^k)$

Proving an asymptotic relationship between two given functions $f(n)$ and $g(n)$ can be done intuitively for most of the functions you will encounter; all polynomials for example. However, this *does not* suffice as a formal proof.

To prove a relationship of the form $f(n) \in \Delta(g(n))$ where Δ is one of $\mathcal{O}, \Omega$, or Θ, can be done simply using the definitions, that is:

- find a value for c (or c_1 and c_2).
- find a value for n_0.

(But this is not the only way.)

Example

Let $f(n) = 21n^2 + n$ and $g(n) = n^3$. Our intuition should tell us that $f(n) \in \mathcal{O}(g(n))$. Simply using the definition confirms this:

$$21n^2 + n \leq cn^3$$

holds for, say $c = 3$ and for all $n \geq n_0 = 8$ (in fact, an infinite number of pairs can satisfy this equation).

Example

Let $f(n) = n^2 + n$ and $g(n) = n^3$. Find a tight bound of the form $f(n) \in \Delta(g(n))$.

Our intuition tells us that

$$f(n) \in \mathcal{O}(n^3)$$

Proof.

$\square$

Proof.

- If $n \geq 1$ it is clear that $n \leq n^3$ and $n^2 \leq n^3$.

$\square$

Proof.

- If $n \geq 1$ it is clear that $n \leq n^3$ and $n^2 \leq n^3$.
- Therefore, we have that

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

□

Proof.

- If $n \geq 1$ it is clear that $n \leq n^3$ and $n^2 \leq n^3$.
- Therefore, we have that

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

- Thus, for $n_0 = 1$ and $c = 2$, by the definition of Big-O, we have that $f(n) \in \mathcal{O}(g(n))$.

$\square$

Example

Let $f(n) = n^3 + 4n^2$ and $g(n) = n^2$. Find a tight bound of the form $f(n) \in \Delta(g(n))$.

Here, our intuition should tell us that

$$f(n) \in \Omega(g(n))$$

Proof.

$\square$

Proof.

- If $n \geq 0$ then

$$n^3 \leq n^3 + 4n^2$$

□

Proof.

- If $n \geq 0$ then
$$n^3 \leq n^3 + 4n^2$$
- As before, if $n \geq 1$,
$$n^2 \leq n^3$$

$\square$

Proof.

- If $n \geq 0$ then

$$n^3 \leq n^3 + 4n^2$$

- As before, if $n \geq 1$,

$$n^2 \leq n^3$$

- Thus, when $n \geq 1$,

$$n^2 \leq n^3 \leq n^3 + 3n^2$$

$\square$

Proof.

- If $n \geq 0$ then
$$n^3 \leq n^3 + 4n^2$$
- As before, if $n \geq 1$,
$$n^2 \leq n^3$$
- Thus, when $n \geq 1$,
$$n^2 \leq n^3 \leq n^3 + 3n^2$$
- Thus by the definition of Big-Ω, for $n_0 = 1, c = 1$, we have that $f(n) \in \Omega(g(n))$.

$\square$

If you have a polynomial of degree 2 such as $an^2 + bn + c$, you can prove it is $\Theta(n^2)$ using the following values:

- $c_1 = \frac{a}{4}$
- $c_2 = \frac{7a}{4}$
- $n_0 = 2 \cdot max(\frac{|b|}{a}, \sqrt{\frac{|c|}{a}})$

Limit Method

Now try this one:

$$\begin{aligned} f(n) &= n^{50} + 12n^3 \log^4 n - 1243n^{12} + \\ &\quad 245n^6 \log n + 12 \log^3 n - \log n \\ g(n) &= 12n^{50} + 24 \log^{14} n^4 3 - \frac{\log n}{n^5} + 12 \end{aligned}$$

Using the formal definitions can be *very* tedious especially when one has very complex functions. It is much better to use the *Limit Method* which uses concepts from calculus.

Limit Method Process

Say we have functions $f(n)$ and $g(n)$. We set up a limit quotient between f and g as follows:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \text{then } f(n) \in \mathcal{O}(g(n)) \\ c > 0 & \text{then } f(n) \in \Theta(g(n)) \\ \infty & \text{then } f(n) \in \Omega(g(n)) \end{cases}$$

- Justifications for the above can be proven using calculus, but for our purposes the limit method will be sufficient for showing asymptotic inclusions.
- Always try to look for algebraic simplifications *first*.
- If f and g *both* diverge or converge on zero or infinity, then you need to apply l'Hôpital's Rule.

l'Hôpital's Rule

Theorem

(l'Hôpital's Rule) Let f and g, if the limit between the quotient $\frac{f(n)}{g(n)}$ exists, it is equal to the limit of the derivative of the denominator and the numerator.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Why do we have to use l'Hôpital's Rule? Consider the following function:

$$f(x) = \frac{\sin x}{x}$$

Clearly, $\sin 0 = 0$ so you may say that $f(x) = 0$. However, the denominator is also zero so you may say $f(x) = \infty$, but both are wrong.

Observe the graph of $f(x)$:

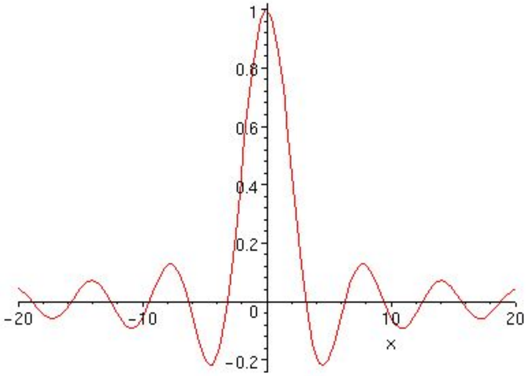


Figure: $f(x) = \frac{\sin x}{x}$

Clearly, though $f(x)$ is undefined at $x = 0$, the limit still exists.

Applying l'Hôpital's Rule gives us the correct answer:

$$\lim_{x \to 0} \frac{\sin x'}{x'} = \frac{\cos x}{1} = 1$$

Example

Let $f(n) = 2^n$, $g(n) = 3^n$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$.

What's our intuition in this case?

Proof.

- We prove using limits.

□

Proof.

- We prove using limits.
- We set up our limit,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{2^n}{3^n}$$

$\square$

Proof.

- We prove using limits.
- We set up our limit,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{2^n}{3^n}$$

- Using l'Hôpital's Rule will *get you no where*:

$$\frac{2^{n\prime}}{3^{n\prime}} = \frac{(\ln 2)2^n}{(\ln 3)3^n}$$

Both numerator and denominator still diverge. We'll have to use an algebraic simplification.

$\square$

Continued.

- Using algebra,

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

Continued.

- Using algebra,

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

- Now we use the following Theorem without proof:

$$\lim_{n\to\infty} \alpha = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

$\square$

Continued.

- Using algebra,

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

- Now we use the following Theorem without proof:

$$\lim_{n\to\infty} \alpha = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

- Therefore we conclude that the quotient converges to zero thus,

$$2^n \in \mathcal{O}(3^n)$$

□

Example

Let $f(n) = \log_2 n$, $g(n) = \log_3 n^2$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$.

What's our intuition in this case?

Proof.

- We prove using limits.

$\square$

Proof.

- We prove using limits.
- We set up our limit,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\log_2 n}{\log_3 n^2}$$

$\square$

Proof.

- We prove using limits.
- We set up our limit,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\log_2 n}{\log_3 n^2}$$

- Here, we have to use the change of base formula for logarithms:

$$\log_\alpha n = \frac{\log_\beta n}{\log_\beta \alpha}$$

$\square$

Continued.

- And we get that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\log_2(n)}{\log_3(n^2)}$$

□

Continued.

- And we get that

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \frac{\log_2(n)}{\log_3(n^2)} \\ &= \frac{\log_2 n}{\frac{2\log_2 n}{\log_2 3}}\end{aligned}$$

$\square$

Continued.

- And we get that

$$\begin{aligned}\lim_{n\to\infty}\frac{f(n)}{g(n)} &= \frac{\log_2(n)}{\log_3(n^2)} \\ &= \frac{\log_2 n}{\frac{2\log_2 n}{\log_2 3}} \\ &= \frac{\log_2 3}{2}\end{aligned}$$

$\square$

Continued.

- And we get that

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \frac{\log_2(n)}{\log_3(n^2)} \\
&= \frac{\log_2 n}{\frac{2\log_2 n}{\log_2 3}} \\
&= \frac{\log_2 3}{2} \\
&\approx .7924\ldots
\end{aligned}$$

$\square$

Continued.

- And we get that

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(n)}{g(n)} &= \frac{\log_2(n)}{\log_3(n^2)} \\
&= \frac{\log_2 n}{\frac{2\log_2 n}{\log_2 3}} \\
&= \frac{\log_2 3}{2} \\
&\approx .7924\ldots
\end{aligned}$$

- So we conclude that $f(n) \in \Theta(g(n))$.

$\square$

Limit Properties

A useful property of limits is that the composition of functions is preserved.

Lemma

For the composition $\circ$ of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then

$$\lim_{n\to\infty} f_1(n) \circ \lim_{n\to\infty} f_2(n) = \lim_{n\to\infty} f_1(n) \circ f_2(n)$$

Useful Identities & Derivatives

Some useful derivatives that you should memorize include

- $(n^k)' = kn^{k-1}$
- $(\log_b(n))' = \frac{1}{n \ln(b)}$
- $(f_1(n)f_2(n))' = f_1'(n)f_2(n) + f_1(n)f_2'(n)$ (product rule)
- $(c^n)' = \ln(c)c^n \leftarrow$ Careful!

Log Identities

- Change of Base Formula: $\log_b(n) = \frac{\log_c(n)}{\log_c(b)}$
- $\log(n^k) = k\log(n)$
- $\log(ab) = \log(a) + \log(b)$

Efficiency Classes

Constant	$\mathcal{O}(1)$
Logarithmic	$\mathcal{O}(\log(n))$
Linear	$\mathcal{O}(n)$
Polylogarithmic	$\mathcal{O}(\log^k(n))$
Quadratic	$\mathcal{O}(n^2)$
Cubic	$\mathcal{O}(n^3)$
Polynomial	$\mathcal{O}(n^k)$ for any $k > 0$
Exponential	$\mathcal{O}(2^n)$
Super-Exponential	$\mathcal{O}(2^{f(n)})$ for $f(n) = n^{(1+\epsilon)}, \epsilon > 0$ For example, $n!$

Table: Some Efficiency Classes

Summary

Asymptotics is easy, but remember:

- Always look for algebraic simplifications
- You *must always* give a rigorous proof
- Using the limit method is always the best
- Always show l'Hôpital's Rule if need be
- Give as simple (and tight) expressions as possible