

Asymptotics

Slides by Christopher M. Bourke
Instructor: Berthe Y. Choueiry

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Computer Science & Engineering 235
Section 2.2 of Rosen
cse235@cse.unl.edu

Notes

Introduction I

Recall that we are really only interested in the *Order of Growth* of an algorithm's complexity.

How well does the algorithm perform as the input size grows;

$$n \rightarrow \infty$$

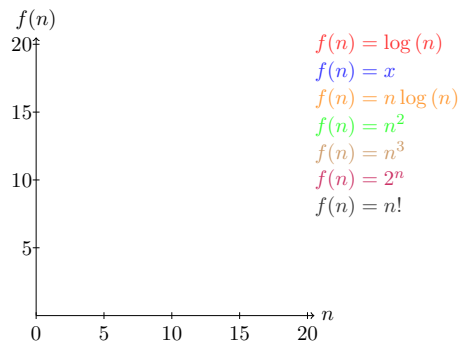
We have seen how to mathematically evaluate the cost functions of algorithms with respect to their input size n and their elementary operation.

However, it suffices to simply measure a cost function's *asymptotic* behavior.

Notes

Introduction

Magnitude Graph



Notes

Introduction I

In practice, specific hardware, implementation, languages, etc. will greatly affect how the algorithm behaves. However, we want to study and analyze algorithms *in and of themselves*, independent of such factors.

For example, an algorithm that executes its elementary operation $10n$ times is better than one which executes it $.005n^2$ times. Moreover, algorithms that have running times n^2 and $2000n^2$ are considered to be *asymptotically equivalent*.

Notes

Big-O Definition

Definition

Let f and g be two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$. We say that

$$f(n) \in \mathcal{O}(g(n))$$

(read: f is Big-"O" of g) if there exists a constant $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$,

$$f(n) \leq cg(n)$$

- ▶ Big-O is actually Omicron, but it suffices to write "O"
- ▶ Intuition: f is (*asymptotically*) less than or equal to g
- ▶ Big-O gives an asymptotic *upper bound*

Notes

Big-Omega Definition

Definition

Let f and g be two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$. We say that

$$f(n) \in \Omega(g(n))$$

(read: f is Big-Omega of g) if there exist $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$,

$$f(n) \geq cg(n)$$

- ▶ Intuition: f is (*asymptotically*) greater than or equal to g .
- ▶ Big-Omega gives an asymptotic *lower bound*.

Notes

Big-Theta Definition

Definition

Let f and g be two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$. We say that

$$f(n) \in \Theta(g(n))$$

(read: f is Big-Theta of g) if there exist constants $c_1, c_2 \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$,

$$c_1g(n) \leq f(n) \leq c_2g(n)$$

- ▶ Intuition: f is (*asymptotically*) equal to g .
- ▶ f is bounded above *and* below by g .
- ▶ Big-Theta gives an asymptotic *equivalence*.

Notes

Asymptotic Properties I

Theorem

For $f_1(n) \in \mathcal{O}(g_1(n))$ and $f_2 \in \mathcal{O}(g_2(n))$,

$$f_1(n) + f_2(n) \in \mathcal{O}(\max\{g_1(n), g_2(n)\})$$

This property implies that we can ignore lower order terms. In particular, for any polynomial $p(n)$ with degree k , $p(n) \in \mathcal{O}(n^k)$.¹

In addition, this gives us justification for ignoring constant coefficients. That is, for any function $f(n)$ and positive constant c ,

$$cf(n) \in \Theta(f(n))$$

Notes

Asymptotic Properties II

Some obvious properties also follow from the definition.

Corollary

For positive functions, $f(n)$ and $g(n)$ the following hold:

- ▶ $f(n) \in \Theta(g(n)) \iff f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$
- ▶ $f(n) \in \mathcal{O}(g(n)) \iff g(n) \in \Omega(f(n))$

The proof is left as an exercise.

¹More accurately, $p(n) \in \Theta(n^k)$

Notes

Asymptotic Proof Techniques

Definitional Proof

Proving an asymptotic relationship between two given functions $f(n)$ and $g(n)$ can be done intuitively for most of the functions you will encounter; all polynomials for example. However, this *does not* suffice as a formal proof.

To prove a relationship of the form $f(n) \in \Delta(g(n))$ where Δ is one of \mathcal{O} , Ω , or Θ , can be done simply using the definitions, that is:

- ▶ find a value for c (or c_1 and c_2).
- ▶ find a value for n_0 .

(But this is not the only way.)

Notes

Asymptotic Proof Techniques

Definitional Proof - Example I

Example

Let $f(n) = 21n^2 + n$ and $g(n) = n^3$. Our intuition should tell us that $f(n) \in \mathcal{O}(g(n))$. Simply using the definition confirms this:

$$21n^2 + n \leq cn^3$$

holds for, say $c = 3$ and for all $n \geq n_0 = 8$ (in fact, an infinite number of pairs can satisfy this equation).

Notes

Asymptotic Proof Techniques

Definitional Proof - Example II

Example

Let $f(n) = n^2 + n$ and $g(n) = n^3$. Find a tight bound of the form $f(n) \in \Delta(g(n))$.

Our intuition tells us that

$$f(n) \in \mathcal{O}(n^3)$$

Notes

Asymptotic Proof Techniques

Definitional Proof - Example II

Proof.

- ▶ If $n \geq 1$ it is clear that $n \leq n^3$ and $n^2 \leq n^3$.
- ▶ Therefore, we have that

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

- ▶ Thus, for $n_0 = 1$ and $c = 2$, by the definition of Big-O, we have that $f(n) \in \mathcal{O}(g(n))$.

□

Notes

Asymptotic Proof Techniques

Definitional Proof - Example III

Example

Let $f(n) = n^3 + 4n^2$ and $g(n) = n^2$. Find a tight bound of the form $f(n) \in \Delta(g(n))$.

Here, our intuition should tell us that

$$f(n) \in \Omega(g(n))$$

Notes

Asymptotic Proof Techniques

Definitional Proof - Example III

Proof.

- ▶ If $n \geq 0$ then

$$n^3 \leq n^3 + 4n^2$$

- ▶ As before, if $n \geq 1$,

$$n^2 \leq n^3$$

- ▶ Thus, when $n \geq 1$,

$$n^2 \leq n^3 \leq n^3 + 3n^2$$

- ▶ Thus by the definition of Big-Ω, for $n_0 = 1, c = 1$, we have that $f(n) \in \Omega(g(n))$.

□

Notes

Asymptotic Proof Techniques

Trick for polynomial of degree 2

If you have a polynomial of degree 2 such as $an^2 + bn + c$, you can prove it is $\Theta(n^2)$ using the following values:

- ▶ $c_1 = \frac{a}{4}$
- ▶ $c_2 = \frac{7a}{4}$
- ▶ $n_0 = 2 \cdot \max\left(\frac{|b|}{a}, \sqrt{\frac{|c|}{a}}\right)$

Notes

Limit Method

Now try this one:

$$\begin{aligned} f(n) &= n^{50} + 12n^3 \log^4 n - 1243n^{12} + \\ &\quad 245n^6 \log n + 12 \log^3 n - \log n \\ g(n) &= 12n^{50} + 24 \log^{14} n^4 3 - \frac{\log n}{n^5} + 12 \end{aligned}$$

Using the formal definitions can be very tedious especially when one has very complex functions. It is much better to use the *Limit Method* which uses concepts from calculus.

Notes

Limit Method Process

Say we have functions $f(n)$ and $g(n)$. We set up a limit quotient between f and g as follows:

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & \text{then } f(n) \in \mathcal{O}(g(n)) \\ c > 0 & \text{then } f(n) \in \Theta(g(n)) \\ \infty & \text{then } f(n) \in \Omega(g(n)) \end{cases}$$

- ▶ Justifications for the above can be proven using calculus, but for our purposes the limit method will be sufficient for showing asymptotic inclusions.
- ▶ Always try to look for algebraic simplifications *first*.
- ▶ If f and g *both* diverge or converge on zero or infinity, then you need to apply l'Hôpital's Rule.

Notes

l'Hôpital's Rule

Theorem

(l'Hôpital's Rule) Let f and g , if the limit between the quotient $\frac{f(n)}{g(n)}$ exists, it is equal to the limit of the derivative of the denominator and the numerator.

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{f'(n)}{g'(n)}$$

Notes

l'Hôpital's Rule I

Justification

Why do we have to use l'Hôpital's Rule? Consider the following function:

$$f(x) = \frac{\sin x}{x}$$

Clearly, $\sin 0 = 0$ so you may say that $f(x) = 0$. However, the denominator is also zero so you may say $f(x) = \infty$, but both are wrong.

Notes

l'Hôpital's Rule II

Justification

Observe the graph of $f(x)$:

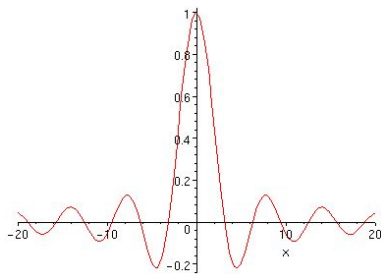


Figure: $f(x) = \frac{\sin x}{x}$

Notes

I'Hôpital's Rule III

Justification

Clearly, though $f(x)$ is undefined at $x = 0$, the limit still exists.

Applying I'Hôpital's Rule gives us the correct answer:

$$\lim_{x \rightarrow 0} \frac{\sin x'}{x'} = \frac{\cos x}{1} = 1$$

Notes

Limit Method

Example 1

Example

Let $f(n) = 2^n$, $g(n) = 3^n$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$.

What's our intuition in this case?

Notes

Limit Method

Example 1 - Proof A

Proof.

- ▶ We prove using limits.
- ▶ We set up our limit,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \frac{2^n}{3^n}$$

- ▶ Using I'Hôpital's Rule will *get you no where*:

$$\frac{2^{n'}}{3^{n'}} = \frac{(\ln 2)2^n}{(\ln 3)3^n}$$

Both numerator and denominator still diverge. We'll have to use an algebraic simplification.

□

Notes

Limit Method

Example 1 - Proof B

Continued.

- ▶ Using algebra,

$$\lim_{n \rightarrow \infty} \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$$

- ▶ Now we use the following Theorem without proof:

$$\lim_{n \rightarrow \infty} \alpha = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

- ▶ Therefore we conclude that the quotient converges to zero thus,

$$2^n \in \mathcal{O}(3^n)$$

□

Notes

Limit Method

Example 2

Example

Let $f(n) = \log_2 n$, $g(n) = \log_3 n^2$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$.

What's our intuition in this case?

Notes

Limit Method

Example 2 - Proof A

Proof.

- ▶ We prove using limits.
- ▶ We set up our limit,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \frac{\log_2 n}{\log_3 n^2}$$

- ▶ Here, we have to use the change of base formula for logarithms:

$$\log_\alpha n = \frac{\log_\beta n}{\log_\beta \alpha}$$

□

Notes

Limit Method

Example 2 - Proof B

Continued.

► And we get that

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} &= \frac{\log_2(n)}{\log_3(n^2)} \\ &= \frac{\log_2 n}{\frac{2 \log_2 n}{\log_2 3}} \\ &= \frac{\log_2 3}{2} \\ &\approx .7924 \dots\end{aligned}$$

► So we conclude that $f(n) \in \Theta(g(n))$. □

Notes

Limit Properties

A useful property of limits is that the composition of functions is preserved.

Lemma

For the composition \circ of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then

$$\lim_{n \rightarrow \infty} f_1(n) \circ \lim_{n \rightarrow \infty} f_2(n) = \lim_{n \rightarrow \infty} f_1(n) \circ f_2(n)$$

Notes

Useful Identities & Derivatives

Some useful derivatives that you should memorize include

- $(n^k)' = kn^{k-1}$
- $(\log_b(n))' = \frac{1}{n \ln(b)}$
- $(f_1(n)f_2(n))' = f_1'(n)f_2(n) + f_1(n)f_2'(n)$ (product rule)
- $(c^n)' = \ln(c)c^n$ ← **Careful!**

Log Identities

- Change of Base Formula: $\log_b(n) = \frac{\log_c(n)}{\log_c(b)}$
- $\log(n^k) = k \log(n)$
- $\log(ab) = \log(a) + \log(b)$

Notes

Efficiency Classes

Constant	$\mathcal{O}(1)$
Logarithmic	$\mathcal{O}(\log(n))$
Linear	$\mathcal{O}(n)$
Polylogarithmic	$\mathcal{O}(\log^k(n))$
Quadratic	$\mathcal{O}(n^2)$
Cubic	$\mathcal{O}(n^3)$
Polynomial	$\mathcal{O}(n^k)$ for any $k > 0$
Exponential	$\mathcal{O}(2^n)$
Super-Exponential	$\mathcal{O}(2^{f(n)})$ for $f(n) = n^{(1+\epsilon)}, \epsilon > 0$ For example, $n!$

Table: Some Efficiency Classes

Notes

Summary

Asymptotics is easy, but remember:

- ▶ Always look for algebraic simplifications
- ▶ You *must always* give a rigorous proof
- ▶ Using the limit method is always the best
- ▶ Always show l'Hôpital's Rule if need be
- ▶ Give as simple (and tight) expressions as possible

Notes
