8.4 Inference for Boolean Theories

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Outline

- Relating domain size and consistency level
- Quick review of basic logic
- Consistency in CNF theories
- Resolution and extended 2-composition
- Interaction graphs
- Directional resolution
- Tractable Boolean Theories

Domain Size & Consistency Level

- Theorem 8.3.2 on page 228 says that strong relational k-consistent networks with domains of size $\leq$ k are globally consistent.

- Thus for k = 2, i.e. bivalued domains such as $\{0,1\}$, enforcing strong relational 2-consistency solves the network.

- For Boolean constraints, relational 2-consistency is enough to achieve global consistency.

- For CNF theories, extended 2-composition is equivalent to pair-wise resolution.

Definitions

Entailment: $\varphi \models \alpha$ means that φ implies α is true in all models of φ.

Prime Implicate: α is a 'prime implicate' of $\varphi \Leftrightarrow \varphi \models \alpha$ and there does *not* exist $\beta \subseteq \alpha$ such that $\varphi \models \beta$. That is, we have removed all clauses that can be subsumed by smaller clauses.

CNF, or Conjunctive Normal Form: $(a \vee b \vee c) \wedge (\neg a \vee b)$ .... A *k-CNF* is a CNF where all clauses have length $\leq$ k.

Horn Form: A *Horn Form* is a CNF form where each clause has at most one non-negated literal, for example, $(\neg a \vee \neg b \vee \ldots \vee z)$.
An equivalent description is an implication: $(\neg a \vee \neg b \vee c) \equiv a \wedge b \Rightarrow c$.

CNF as CSP

CNF theories can be modeled as CSPs, with each literal a variable, each clause a relation.

Example $(F \vee X \vee Y \vee \neg Z) \wedge (F \vee Z) \wedge (F \vee X \vee Y)$

Let's draw the network:

What do we see in such a network?

- All clauses are relationally arc-consistent — projection of R_i doesn't prune either 0 or 1 from any domain.

- Clauses 1 and 2 are relationally path-consistent relative to F — any truth assignment to X, Y and Z can be extended by assigning TRUE to F, satisfying both relations.

- The same clauses are relationally path-consistent relative to Z — any consistent assignment to X, Y and F can be extended by assigning TRUE to Z.

- In fact, the entire set of clauses is relationally path-consistent.

Resolution and Extended 2-composition

Lemma 8.4.3: Resolving $(\alpha \vee Q)$ and $(\beta \vee \neg Q)$ results in $(\alpha \vee \beta)$. The equivalent result in the language of CSPs is:
$models(\alpha \vee \beta) = EC_{\alpha,\beta}(models(\alpha \vee Q), models(\beta \vee \neg Q))$.

Remember, to get $EC_A(R_{S_1}, \ldots, R_{S_m})$, join all relations R_{S_i}, pick a subset A of the scopes, and project the joined relations over A (p. 220).

Thus, we can modify algorithm RC_1 (page 220), replacing line 4 with the resolution rule. The result adds/updates binary resolvent relations until no more can be added.

Why does this matter?

Because enforcing relational 2-consistency in a bi-valued network generates a backtrack-free network, we can solve the network with this algorithm.

Cleaning up the Representation

Subsumption elimination is where if $\alpha \subseteq \beta$ and we find α in the theory, β is implied, so we can ignore β without losing information.

Now if we run the modified RC_2 <u>and</u> subsumption elimination, the result is all the prime implicates of the theory, which are globally consistent.

Interaction Graphs

Given a theory φ, G(φ) is an undirected graph with one node per variable, and one edge for each pair of nodes found in the same clause. That is, each clause becomes a clique.

Let's draw G(φ) where $\varphi = \{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}$:

Interaction Graphs (cont'd)

Now let's perform resolution over A:

- Pick two clauses containing A and ¬A: $(A \vee B \vee C)$ and $(\neg A \vee B \vee E)$

- Resolve away A, creating a clause over the other variables in the two clauses: $\text{res}((A \vee B \vee C), (\neg A \vee B \vee E)) = (B \vee C \vee E)$.

- Draw the edge(s) representing the new resolvent:

How does this relate?

- CNF domain constraints correspond to unit literals.

- Thus, Relational Arc Consistency (Eq. 8.7) corresponds to unit-resolution.

- The UNIT-PROPAGATION algorithm of Figure 3.16 uses relational arc-consistency when working on SAT problems.

- Unlike Relational Arc Consistency, unit propagation can be done in linear time.

Directional Resolution

Bucket processing algorithm DRC_2 can be modified for CNF networks by replacing extended 2-composition with boolean resolution, and instantiation with unit resolution.

The result assembles a backtrack-free search for a given ordering, if no empty clauses are generated at any step.

Directional Resolution Example (pp. 232-233)

Let $\varphi =$
$\{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}$
Let ordering $d = (E, D, C, B, A)$.

1. Draw the interaction graph G(φ) along ordering d.

2. Fill the buckets with the initial constraints.

3. Apply resolution from latest to earliest, adding new resolvents to bucket of latest variable in the scope. Induce the corresponding arcs.

DRC Complexity

Lemma 8.4.5 Given a theory φ and and ordering d = $(Q_1, \ldots, Q_n)$, if Q_i has at most k parents in the induced graph along d, then the bucket of Qi in $E_d(\varphi)$ contains no more than 3^{k+1} clauses.

Because the number of parents each variable can have is bounded by the induced width along the chosen ordering, we get:

Theorem 8.4.6 Given a theory φ and and ordering d = $(Q_1, \ldots, Q_n)$, the time complexity of algorithm DR along d is $O(n \cdot 9^{w_d^*})$, and $E_d(\varphi)$ contains at most $n \cdot 3^{w_d^*+1}$ clauses, where w_d^* is the induced with of φ's interaction graph along d.

Tractable Boolean Theories

2-SAT

- 2-CNF theories are closed under resolution — resolvents are of size 2 or less.

- The overall number of clauses of size 2 is bounded by $O(n^2)$

- Therefore, the algorithm just shown (DR) is tractable for 2-CNF theories.

- However, standard 2-SAT algorithms are still the fastest.

Horn Theories

- Horn theories are also tractable.

- Unit propagation produces unit clauses and non-unit clauses, and an empty clause if the problem is unsatisfiable (Theorem 8.4.8).

- To get a solution from the result, set variables in unit clauses to true values, and all other variables to false.

- Unit-propagation is the same as relational arc consistency. Thus, either can solve a Horn theory.