Scribe notes on the class discussion on consistency methods for boolean theories, row convex constraints and linear inequalities (Section 8.3 to 8.6)

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1 Inference for Boolean theories

In this part of the class we discussed the effect of domain size on the consistency required for BT-free search and its applicability to boolean theories. The notion of equivalence of resolution to extended 2-composition (seen in Chapter 8, Section 8.1 from [1]) was studied. We saw how DRC_2 can be specialized for CNF theories, and also some tractable boolean theories.

1.1 Discussion on effect of domain size on consistency level

- Strong relational k-consistent networks with domain size $\leq k$ are globally consistent (Theorem 8.3.2 on page 228 of [1])
- For boolean constraints, the domain size of variables is 2. Thus enforcing relational 2-consistency is sufficient for global consistency (page 228 of [1]).
- Boolean domains are not restricted to $\{True, False\}$ but they can have any domain, such as $\{Y, N\}$ or $\{0, 1\}$, that are a homomorphism to the $\{True, False\}$ domain, as long as the domain is bivalued.

1.2 Consistency for CNF theories

- Before beginning the discussion on the topic, the following definitions were reviewed:
  1. Entailment
  2. Prime implicate
  3. Conjunctive Normal Form (CNF)
  4. Horn clauses
- CNF theories can be modeled as CSPs as follows,
  - Each literal corresponds to a variable
  - Each clause to a constraint
  - The domain of a each variable is $\{True, False\}$
- Consider two clauses $(\alpha \vee Q)$ and $(\beta \vee \neg Q)$. The resolution of the two clauses gives us $(\alpha \vee \beta)$. Lemma 8.4.3 in [1] says that the models($\alpha \vee \beta$) can be obtained by doing extended composition of models($\alpha \vee Q$) and models($\alpha \vee \neg Q$) relative to α and β. Since the composition is over two variables it is Extended 2-composition.

- Using this result we can specialize algorithm RC_2 (from Chapter 8, section 8.1 of [1] for performing relational 2-consistency) for CNF theories. This is done by replacing extended 2-composition with resolution. As we saw above these are equivalent. The algorithm will give a CNF theory CSP which is relational 2-consistent.

- As seen earlier in the section, relational 2-consistent networks with domain size of 2 are globally consistent. Hence applying the resolution algorithm generates a globally consistent and hence backtrack-free representation.

- Next we saw the definition of subsumption elimination: If we have $\alpha \subset \beta$, and we find α in the theory, then β is implied. So we can eliminate β. This elimination is called subsumption elimination.

- If we run RC_2 (with resolution) and also subsumption elimination, the result is all the prime implicates of the theory, which are globally consistent.

- We then looked at the definition of an interaction graph of a given theory. On performing resolution over a theory, its interaction graph is also updated to reflect the added resolvents.

- Since a domain constraint in CNF corresponds to a unit literal, the relational arc-consistency rule (Equation 8.7, on page 221 of [1]) corresponds to unit resolution.

- The UNIT-PROPAGATION algorithm (shown in Fig 3.16 on page 85 of [1]) applies relational arc-consistency for SAT problems. But it is to be noted that unit-propagation can be done in linear time, unlike relational arc-consistency.

1.3 Directional Resolution

- The class then discussed how specializing DRC_2 (page 222, Figure 8.2 of [1]) for CNFs by replacing extended 2-composition with resolution and the instantiation step by unit resolution, results in algorithm DIRECTIONAL-RESOLUTION or DR (Section 8.4 page 232 of [1]).

- The space and time complexity of directional-resolution is exponentially bounded in the *induced width* of the theory's interaction graph along the order *d*.

1.4 Tractable boolean theories

- The class then discussed two boolean theories, for which the DR algorithm is tractable. The two class of problems that are tractable using the DR algorithm are:

  1. 2-SAT
  2. Horn theories

- 2-SAT: 2-CNF theories are closed under resolution (the resolvents are of size 2 or less). The overall number of clauses of size 2 is bounded by $O(n^2)$. Therefore DR is tractable for 2-SAT.

- Directional-resolution is not the best way to solve 2-SAT, standard 2-SAT algorithms solve them in linear time.

- Horn theories: Unit propagation produces unit clauses and non-unit clauses, and an empty clause if the problem is unsatisfiable. To get a result, set variables in the unit clauses to true values and all others to false. We saw that unit propagation is the same as relational arc-consistency and hence we conclude that Horn theories are tractable.

2 Row convex constraints

We then moved on to discuss a property of constraints called row convexity and show that for row convex constraints relational path consistency is sufficient to guarantee global consistency.

2.1 Row convexity

- Row convexity extends some special binary constraints such as functional and monotone constraints. These constraints were defined first. For these properties we viewed a binary constraint as a matrix (see Chapter 1 page 26 of [1] for details).
- Functional: A binary relation R_{ij} expressed as a (0,1) matrix is functional iff there is at most a single "1" in each row and in each column of R_{ij}
- Monotone: In simple words in a constraint matrix R_{ij} everything to the left of a 1 is 1 and everything to bottom of a 1 is 1. Given some ordering of the domain of values for all variables, a binary relation R_ij expressed as a (0,1) matrix is monotone if the following conditions hold: if *(a,b)* $\in R_{ij}$ and $c \geq a$, then *(c,b)* $\in R_{ij}$, and if *(a,b)* $\in R_{ij}$ and $c \leq b$, then *(a,c)* $\in R_{ij}$.
- Row convex constraints: A binary relation R_{ij} represented as a (0,1)-matrix is row convex if in each row (column) all of the ones are consecutive; that is, no two ones within a single row are separated by a zero in that same row (column).
- Arc consistency in a CSP can be verified by looking at the constraint matrices. If every constraint matrix has a "1" in every row then we ensure arc-consistency.
- Lemma 8.5.4 is on row convexity and intersection is a discrete version of the well-known result called Helly's theorem (for details see [2]).
- Let F be a finite collection of (0,1)-row vectors that are row convex and of equal length such that every pair of row vectors in F have a non-zero entry in common; that is, their intersection is not the vector with all zeroes. Then all of the row vectors in F have a non-zero entry in common.
- Theorem 8.5.5 says that if we can find an order of domains such that the relations of a path consistent CSP R are row convex, then the network is globally consistent and is therefore minimal.
- Dr. Choueiry brought to notice the point that finding an order of domains to get row convexity can be done in polynomial time.
- The theorem becomes apparent by observing that path consistency forces all rows to share a common non-zero entry with some other row and row convexity forces all those rows to share the same entries or entry. All the relations $R_{i,k}$ sharing a non-zero entry means that there is some value in the domain of k which is consistent with all the relations among all the relations between the k domains. The CSP is hence k-consistent and as k has an upper bound of n, the CSP is consistent.
- It is to be noted that row-convexity and path consistency guarantee a solution, but existence of a solution does not guarantee row convexity.
- Another point to keep in mind is that the constraints should be row-convex after path consistency is applied. Row convex constraints may not remain so after making them path consistent.

2.2 Identifying row convex relations

An $m \times n$ (0,1)-matrix with f non-zero entries can be tested for whether a permutation of the columns exists such that the matrix is row-convex in *O(m+n+f)* steps.

2.3 Non-binary row-convex constraints

- An r-ary relation is row convex if, in the multidimensional 0/1 matrix representing the constraint, each vector that is parallel to one of the axes has the consecutive 1's property.
- Theorem 8.5.10, says that general relational path consistency ensures global consistency when the relations are row convex.
- Theorems 8.5.11 through 8.5.13 are specialized versions of theorem 8.5.10. Theorem 8.5.11 says that RC_2 will give a globally consistent network if the closure under extended 2-composition is row convex. Theorem 8.5.12 is a specialization for linear inequalities. Theorem 8.5.13 extends the idea to directional- relational consistency (using DRC_2).

3 Linear inequalities

Here we looked at the application of consistency enforcing techniques to linear inequalities over infinite domains like rational numbers, and also infinite and finite subsets of integers.

3.1 Linear elimination

- Linear inequality: We consider constraints between r or fewer variables having the form $\sum_{i=1}^{r} a_i xi \leq c$, where a_i and c are rational constants.
- The following is a step wise "proof" presented by Eric in class to show that RC_2 or DRC_2 solve linear inequalities.
  - A linear inequality cuts the variable space into a half space.
  - Half spaces are convex
  - Intersection of half spaces is convex
  - Hence a system of linear inequalities is convex
  - Convex regions with rational domains are row convex
  - Thus $(D)RC_2$ solves the system of linear inequalities
- Linear elimination: Consider two linear inequality constraints $\alpha = \sum_{i=1}^{(r-1)} a_i x_i + a_r x_r \leq c$ and $\beta = \sum_{i=1}^{(r-1)} b_i x_i + b_r x_r \leq d$. We can perform linear elimination of variable x_r only if a_r and b_r are of opposite signs. Linear elimination can be performed by making the magnitude of the co-efficients of x_r same but keeping the signs opposite and adding α and β. If a_r and b_r have the same sign, the elimination implicitly generates the universal constraint. In the latter case, elimination does not lead to any gain in information.
- If the domains of variables are finite, we can convert each linear inequality to a relational representation and then enforce relational path-consistency to get a solution. In case the domains are infinite or as an alternative we can perform linear elimination directly to algebraic expressions.

- We saw that linear elimination is identical to extended 2-composition over rational domains, but not over integer domains. The class discussed the reason for the loss of some solutions of linear inequalities with integer domains. The reason for the "loss" of solutions when domains are integers is due to the fact that some solution-point may lie in between integer values. In other words the discretization of the integer domain leads to a few solutions to slip away.

3.2 Fourier bucket elimination

- By incorporating linear elimination into DRC_2, when the constraints are specified as linear inequalities yields the algorithm Directional Linear Elimination (shown in figure 8.13 of [1]), which is the Fourier elimination algorithm.

- The algorithm takes as input a set of linear inequalities and an ordering d, and decides the solvability of the linear inequalities and generates a backtrack-free problem representation.

- The algorithm partitions the inequalities into buckets based on the ordering d. Then it processes the buckets in reverse order of d. For each bucket, the algorithm performs linear elimination between all possible pairs of linear inequalities. The new linear inequality produced is placed in an appropriate bucket. If a linear inequality which has no solutions is encountered, inconsistency is reported and the algorithm terminates. In case there is only one in the domain of a variable,, the algorithm substitutes its value in the inequalities of the bucket and places the resulting linear inequalities in appropriate buckets. After all buckets have been processed, the algorithm guarantees a backtrack-free instantiation of variables in the remaining linear inequalities.

References

[1] Rina Dechter. Constraint Processing. Manuscript, forthcoming 2002, 2002.

[2] J Eckhoff, P M Gruber, and J M Wills. *Handbook of Convex Geometry*, chapter 2.1, pages 389–448. Elsevier Science Ltd, Amsterdam Netherlands, 1993.