Hash tables (II)

Textbook, Chapter 12, Sections 12.3 and 12.4 (partially)

CSCE310: Data Structures and Algorithms

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Hash functions

A good hash function

- can be computed quickly
- satisfies (approx.tely) the simple uniform hashing assumption: each key is equally likely to hash to any of the m slots

Formally:
Assume each key is drawn independently from U with probability distribution P
$P(k)$ is the probability that k is drawn
Simple uniform hashing $\Rightarrow$

$$\sum_{k:h(k)=j} P(k) = \frac{1}{m}, \quad \text{for} \;\; j = 0, 1, \ldots, m-1$$

However, P is usually unknown

Formally: Simple uniform hashing requires

$$\sum_{k:h(k)=j} P(k) = \frac{1}{m}, \quad \text{for} \;\; j = 0, 1, \ldots, m-1$$

Problem: we rarely know distribution $P(k)$

When keys are random real numbers k independently and **uniformly** distributed in the range of $0 \leq k < 1$,

$$h(k) = \lfloor km \rfloor$$

satisfies simple uniform hashing assumption

Hash functions

1. Division method
2. Multiplication method
3. Universal hashing

Assumption: keys are natural numbers ($\in \mathbb{N}$)

If keys $\notin \mathbb{N}$, find a way to express them as such

Example: strings can be interpreted by interpreting each character as an integer in notation radix-128, using the ASCII character set
Illustration: `pt` can be interpreted as $(112.128) + 116 = 14452$, since $p = 112$ and $t = 116$ in the ASCII character set

In radix-2: 1 digit, 2 possibilities

In a word of k digits (bits), there are 2^k possibilities

$a = \langle a_0, a_1, a_2, \ldots, a_{k-1} \rangle$

$a_{k-1}2^{k-1} + a_{k-2}2^{k-2} + \ldots + a_2 2^2 + a_1 2^1 + a_0 2^0 \equiv$

$a_{k-1}2^{k-1} + a_{k-2}2^{k-2} + \ldots + a_2 2^2 + a_1 2 + a_0$

(binary, $a_i \in \{0, 1\}$) largest term is $2^k - 1$

In radix-10: 1 digit, 10 possibilities

In a word of k digits, there are 10^k possibilities

$a = \langle a_0, a_1, a_2, \ldots, a_{k-1} \rangle$

$a_{k-1}10^{k-1} + \ldots + a_2 10^2 + a_1 10^1 + a_0 10^0 \equiv$

$a_{k-1}10^{k-1} + \ldots + a_2 10^2 + a_1 10 + a_0$

(decimal, $a_i \in \{0, 1, \ldots, 9\}$)

largest term is $10^k - 1$

In radix-16: 1 digit, 16 possibilities

In a word of k digits, there are 16^k possibilities

$a = \langle a_0, a_1, a_2, \ldots, a_{k-1} \rangle$

$a_{k-1}16^{k-1} + \ldots + a_2 16^2 + a_1 16^1 + a_0 16^0 \equiv$

$a_{k-1}16^{k-1} + \ldots + a_2 16^2 + a_1 16 + a_0$

(hexadecimal, $a_i \in \{0, 1, \ldots, 9, A, B, \ldots, F\}$)

largest term is $16^k - 1$

In radix-128: 1 digit, 128 possibilities

In a word of k digits, there are 128^k possibilities

$a = \langle a_0, a_1, a_2, \ldots, a_k \rangle$

$a_{k-1}128^{k-1} + \ldots + a_2 128^2 + a_1 128^1 + a_0 128^0 \equiv$

$a_{k-1}128^{k-1} + \ldots + a_2 128^2 + a_1 118 + a_0$

(ASCII, $a_i \in \{0, \ldots, 127\}$)

Example: `pt` is $112 \times 128 + 116 = 14452$ (bcz $p = 112$, t=116)

largest term is $128^k - 1$

Notes

$1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$

biggest digit = B. Value of biggest digit = x-1 , radix-x
$B + Bx^2 + Bx^3 + \ldots + Bx^{k-1} = B\frac{(x^k - 1)}{x - 1}$

value of biggest term in a word of k-digits in radix x is $(x^k - 1)$

Notes

Inversely, to represent n possibilities in radix-k, we need $\log_k n$ digits

To represent 4 possibilities in radix-2, we need $\log_2 4 = 2$ digits

To represent 128 possibilities in radix-2, we need $\log_2 128 = 8$ digits

To represent 10 possibilities in radix-10, we need $\log_{10} 10 = 1$ digit

To represent 100 possibilities in radix-10, we need $\log_{10} 100 = 2$ digits

Binary coding

A word of k bits in radix-2 has 2^k possibilities

To put in radix-2^l (binary coding) we need $\log_{2^l} 2^k = \frac{\lg 2^k}{\lg 2^l} = \frac{k}{l}$ (binary) digits, k words of l bits

Division method (I)

Principle:

takes the remainder of k divided by m: $h(k) = k \bmod m$

Example:

$m = 12$ and $k = 100 \Rightarrow h(k) = 8 \times 12 + 4 \pmod{12} \equiv 4 \pmod{12}$

Characteristics:

- Quick hash function
- Avoid certain values of m:
  1. k is a binary number, avoid $m = 2^p$
  2. k is a decimal number, avoid $m = 10^p$
  3. k is a character string in radix 2^p, avoid $m = 2^p - 1$
  4. Good values for m are primes not too close to exact powers of 2
     Example: $n = 2000$ character strings
     If $\alpha = 3$ then choose $m = 701$, a prime not too close to a power of 2 (512 and 1024) $\Rightarrow h(k) = k \bmod 701$

Values of m to avoid: m power of 2, $m = 2^p$

Consider the key a, a binary number,
$a = \langle a_0, a_1, \ldots, a_k \rangle = a_k 2^k + a_{k-1} 2^{k-1} + \ldots + a_1 2 + a_0$

Assuming $k > p$, the key can be written
$a = 2^p(a_k 2^{k-p} + \ldots + a_p) + (a_{p-1} 2^{p-1} + \ldots + a_1 2 + a_0)$
where $(a_{p-1} 2^{p-1} + \ldots + a_1 2 + a_0) < 2^p$
thus $a = q 2^p + r$
where $q = a_k 2^{k-p} + \ldots + a_p$ and $r = a_{p-1} 2^{p-1} + \ldots + a_1 2 + a_0$
and $h(a) = a \bmod m = a \bmod 2^p = q 2^p + r \bmod 2^p \equiv r$, where
$r = \langle a_{p-1}, \ldots, a_1, a_0 \rangle$

If $m = 2^p$, then $h(k)$ is just the p lowest order bits of k

Values of m to avoid: m power of 2, $m = 2^p$

If $m = 2^p$, then $h(k)$ is just the p lowest order bits of k

Lesson: Unless $P(k)$ makes all low-order p-bits patterns equally likely, it is preferable to make the hash function depend on all the bits of the key

Values of m to avoid: m power of 10, $m = 10^p$

Similarly, avoid powers of 10 if keys are decimal numbers (as hash function does not depend on all the decimal digits of k)

Values of m to avoid: $m = 2^p - 1$ (I)

when the key is a character string in radix 2^p, two strings that are identical except for a transportation of two adjacent characters will hash to the same value

We prove for two keys a, b, such that:
$a = \langle a_0, a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k \rangle$
$b = \langle a_0, a_1, \ldots, a_j, \ldots, a_i, \ldots, a_k \rangle$

we have $h(a) = h(b)$

Values of m to avoid: $m = 2^p - 1$ (II)

$a = \langle a_0, a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k \rangle$
$b = \langle a_0, a_1, \ldots, a_j, \ldots, a_i, \ldots, a_k \rangle$ then
$h(a) = (a_k 2^{kp} + \ldots + a_i 2^{ip} + \ldots + a_j 2^{jp} + \ldots + a_1 2^p + a_0) \bmod (2^p - 1) =$
$q(2^p - 1) + r \bmod (2^p - 1)$ and
$h(b) = (a_k 2^{kp} + \ldots + a_j 2^{ip} + \ldots + a_i 2^{jp} + \ldots + a_1 2^p + a_0) \bmod (2^p - 1) =$
$q'(2^p - 1) + r' \bmod (2^p - 1)$

$\Rightarrow h(a) - h(b) = a_i 2^{ip} + a_j 2^{jp} - a_j 2^{ip} - a_i 2^{jp} = (q - q')(2^p - 1) + r - r'$
$\Rightarrow a_i(2^{ip} - 2^{jp}) - a_j(2^{ip} - 2^{jp}) = q''(2^p - 1) + r - r'$
$\Rightarrow a_i 2^{jp}(2^{(i-j)p} - 1) - a_j 2^{ip}(2^{(i-j)p} - 1) = q''(2^p - 1) + r - r'$
$\Rightarrow (a_i 2^{jp} - a_j 2^{ip})(2^{(i-j)p} - 1) = q''(2^p - 1) + r - r'$
However, $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}) \Rightarrow$
$(2^{(i-j)p} - 1) = (2^p - 1)(\ldots) = (2^p - 1)A$
$\Rightarrow (a_i 2^{jp} - a_j 2^{ip})(2^p - 1)A = q''(2^p - 1) + r - r'$
$\Rightarrow r - r' = 0$
$\Rightarrow h(a) = h(b)$ *q.e.d*

Multiplication method

Use: $h(k) = \lfloor m(kA \bmod 1) \rfloor$, where $kA \bmod 1 = ka - \lfloor kA \rfloor$

Two steps:

1. Multiply k by A constant $(0 < A < 1)$, and extract the fractional part of kA
2. Then multiply value by m and take floor of result

Characteristics:

- Value of m is not critical
- We can choose m to make h-function easy to implement
  Choose for $m = 2^p$ (bcz multiplication by m would correspond to a simple left shifting of p positions, shifting instructions available on most hardware)
- Knuth suggests using $A \approx \frac{(\sqrt{5}-1)}{2}$

Multiplication method: an example

$$h(k) = \lfloor m(kA \bmod 1) \rfloor$$

Suppose $k = 123456, m = 10000, A = 0.6180339887$ *decimal numbers*

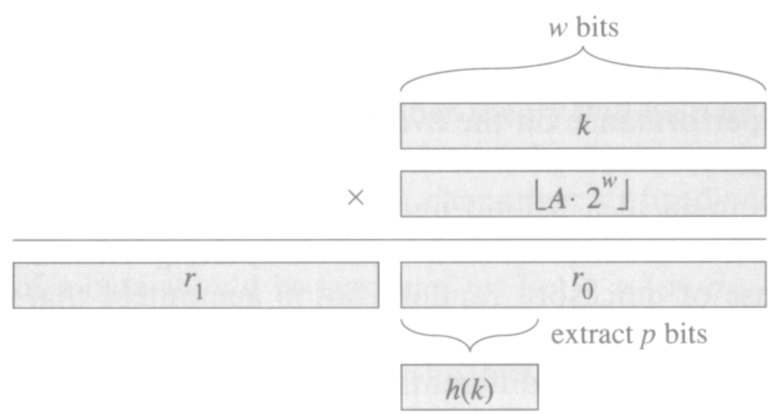
$$\begin{aligned} h(k) &= \lfloor 10000 \cdot (123456 \cdot 0.61803\ldots \bmod 1) \rfloor \\ &= \lfloor 10000 \cdot (76300.0041151\ldots \bmod 1) \rfloor \\ &= \lfloor 41.151\ldots \bmod 1) \rfloor \\ &= 41 \end{aligned}$$

Do exercise 12.3.4

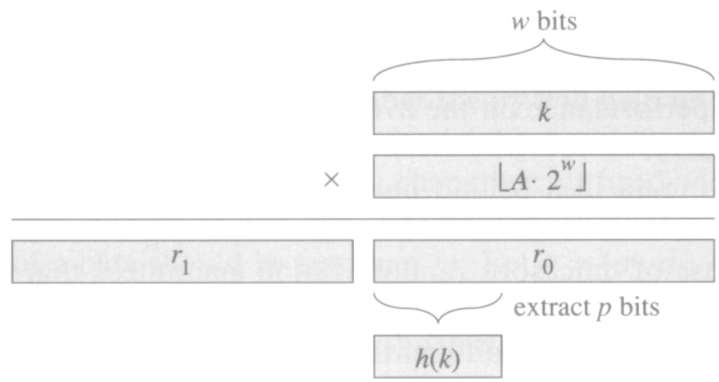
Multiplication method: an implementation (I)

Situation:

- word size of the machine is w-bits
- keys are binary numbers, fit into 1 word
- no multiplication of rational rational numbers



Multiplication method: an implementation (II)



Steps:

- $0 < A < 1$ and no multiplication of rational rational numbers $\rightarrow$ use $\lfloor A.2^w \rfloor$
- Multiply k and $\lfloor A.2^w \rfloor \rightarrow$ (each of w bits) yields a word of $2w$-bits of value $r_1 2^w + r_0$
- r_0 is the fractional part of kA (approx)
- Multiplying by $m = 2^p$ corresponds to taking the p most significant bits of r_0

Universal hashing

Worst-case scenario:

- Malicious adversary chooses the keys to be hashed
- Bad choice of hashing, all n keys hash to the same slot

Average retrieval time deteriorates: $\Theta(n)$

Any *fixed* hash function is vulnerable

Way out?

Choose a h-function that is <u>random</u>

independent of keys to be stored

The scheme is called **<u>universal hashing</u>**

yields good performance on average

no matter what keys are chosen by adversary

Universal hashing: principal

Select a hash function at random, at run time from a carefully designed class of functions

Randomization guarantees

- that no single input will evoke worst-case behavior
- good average-case performance, no matter what keys are provided as input

Universal hashing: principal

Let $\mathcal{H}$ be a finite set of hash functions that map a universe U of keys into the range $\{0, 1, 2, \ldots, m-1\}$

$\mathcal{H}$ is <u>universal</u> if for every pair of distinct key $x, y, \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(x) = h(y)$ is precisely $|\mathcal{H}|/m$

that is, with a hash function randomly chosen from $\mathcal{H}$, the chance of collision between x and y when $x \neq y$ is exactly $1/m$, which is exactly the chance of a collision if $h(x)$ and $h(y)$ are randomly chosen from the set $\{0, 1, 2, \ldots, m-1\}$

Theorem 12.3: If h is chosen from a universal collection of hash functions and is used to hash n keys into a table of size m, where $n \leq m$, the expected number of collisions involving a particular key k is less than 1

Example of A universal class of hash functions

Principle

- Goal: Create a universal class of functions that hash a key into a value between 1 and m-1
- Proposal: each function is defined for an index a, such that a is as a sequence $\langle a_0, a_1, \dots, a_r \rangle$ where each $a_i < m$

  Each key x we receive is represented as a sequence $\langle x_0, x_1, \dots, x_r \rangle$ where each $x_i < m$

  The hash function is $h_a(x) = \sum_{i=0}^{r} a_i x_i \bmod m$

Example of A universal class of hash functions

Implementation

- Choose m, table size, to be a prime number
  reasons: (1) remember, divi!sion method!
  (2) prove collision probability is $1/m$

- Choose $(r+1)$ a_i's, each a_i chosen randomly from $\{0, 1, \ldots, m-1\}$
  There are $m^(r+1)$ **such possibilities**
  This yields $a = \langle a_0, a_1, \ldots, a_r \rangle$
  Now a is fixed

- When user gives the key x, decompose x in $r+1$ parts of $x = \langle x_0, x_1, \ldots, x_r \rangle$

- Compute $h_a(x) = \sum_{i=0}^{r} a_i x_i \bmod m$

- We can prove that $\mathcal{H} = \bigcup_a \{h_a\}$ is a universal class of hash functions
  Since there are $m^(r+1)$ possibilities for a, there are $m^(r+1)$ members in the class

Proof: $\mathcal{H} = \bigcup_a \{h_a\}$ is a universal class of hash functions

Given x, y, two distinct keys, prove that $h_a(x) = h_a(y)$ with probability $1/m$

- Consider x, y, two distinct keys. For example, $x_0 \neq y_0$, and rest x_i, y_i are the same
- $h_a(x) = h_a(y) \Rightarrow \sum_{i=0}^{r} a_i x_i = \sum_{i=0}^{r} a_i y_i \bmod m$
  $\Rightarrow a_0 x_0 + \sum_{i=1}^{r} a_i x_i = a_0 y_0 + \sum_{i=1}^{r} a_i y_i \bmod m$
  $\Rightarrow a_0(x_0 - y_0) = -\sum_{i=1}^{r} a_i(x_i - y_i) \bmod m$
  but m is prime $\Rightarrow (x_0 - y_0)$ has an inverse for multiplication mod$m$ (multiplication mod$m$ is a finite field, Galois field)
  $\Rightarrow a_0 = -\frac{\sum_{i=1}^{r} a_i(x_i - y_i)}{(x_0 - y_0)} \bmod m$
- There are m^r such values for a_0
- x and y collide once for each value of such value of a_0
  $\Rightarrow x$ and y collide with probability $\frac{m^r}{m^{r+1}} = 1/m$ *q.e.d*

Open addressing

- All elements are stored in hash table
  Each table entry contain either an element or NIL
- Search considers systematically (but not linearly) table slots
  until element is found or it becomes clear element is not in table
- Sequence of slots to be examined is <u>**computed**</u>
- no lists, no elements stored outside table
  $\Rightarrow$ When the table is "filled up", no element can be inserted
  $\Rightarrow$ Load factor $\alpha < 1$
- Advantage: avoids pointers.
  Extra memory freed by pointers can be used to increase the number of slots in table

Insertion in open addressing

Principle

- Successively examine or **probe** hash table until we find an empty slot
- Slots are not visited linearly, $\Theta(n)$ search time for empty slot, positions probed are computed from key to be inserted
- Hash function includes the probe number i, $h(k) \rightarrow h(k,i)$

$$h : U \times \{0,1,2,\ldots,m-1\} \longrightarrow \{0,1,2,\ldots,m-1\}$$

- For every key k, the **probe sequence** $\langle h(k,0), h(k,1), \ldots, h(k,m-1)\rangle$ **must be a permutation** of $\langle 0,1,\ldots,m-1)\rangle$ (so that every position is eventually considered as a slot for a new key as table fills up)

Insertion in open addressing

Assumption: elements are keys with no satellite data

```
HASH-INSERT(T, k)
i ← 0
repeat j ← h(k, i)
        if T[j] = NIL
          then T[j] ← k
                 return j
          else i ← i + 1
  until i = m
error "hash table overflow"
```

Each slot contains a key or Nil

Searching in open addressing

`Hash-Search(T, k)` probes same sequence of slots that `Hash-Insert(T,k)` examined when inserting the key

```
HASH-SEARCH(T, k)
i ← 0
repeat j ← h(k, i)
        if T[j] = k
           then return j
        i ← i + 1
  until T[j] = NIL or i = m
return NIL
```

Returns j if slot j contains key k, or NIL if key k is not present in T

When search finds an empty slot, search stops (k would have been inserted there and not later), assuming keys are not deleted from hash table

Deletion in open addressing is difficult

Solution:

- When deleting a key from slot i, don't put Nil, because we won't be able to search for another key for which we probed slot i, found it busy. Instead mark slot with value `Deleted` instead of `Nil`
- In this case, `Hash-Search` should keep looking when t when it finds `Deleted`, while `Hash-Search` treats such a slot as if it was empty