Hash tables

Textbook, Chapter 12, Sections 12.1, 12.2, 12.3

CSCE310: Data Structures and Algorithms
`www.cse.unl.edu/~choueiry/S01-310/`

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Dynamic set (i.e., dictionary) operations required by many applications:

1. `Insert`
2. `Search`
3. `Delete`

$\longrightarrow$ A hash table is an effective data structure
basic operations in $O(1)$ on average

Worst case: as bad as a linked list ($\Theta(n)$)
In practice: extremely competitive (nearly constant)
basic operations in $O(1)$ on average

Hash table: A generalization of notion of an ordinary array

Array uses direct addressing, which

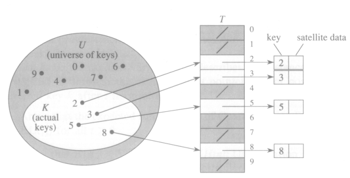
- allows access to an arbitrary position in $O(1)$
- requires one position for every possible key

Hash table

- does not use key as array index, but *computes* array index from key
- is advantageous when #keys actually stored « #keys possible
- uses an array of size proportional to # of keys stored
- object can be stored in slot itself (instead of pointer)

Direct-access tables= array, $T[0, 1, \ldots, m-1]$

Each element has a key drawn from $U = \{0, 1, \ldots, m-1\}$
Assumption: no two elements have the same key



- Slot k points to the element in set with key k
- When no element in set with key$= k$, $T[k] =$ `Nil`
- Works well when U, universe of keys, is small

Dictionary operations *Trivial*

`Direct-Address-Search` (T, k)
return $T[k]$

`Direct-Address-Insert` (T, x)
return $T[key[x]] \leftarrow x$
takes as input pointer to x, not the key

`Direct-Address-Delete` (T, x)
return $T[key[x]] \leftarrow Nil$
takes as input pointer to x, not the key

$\longrightarrow O(1)$

Direct-access table

- When `U` is large, storing T of size `U` is impractical
- When `k` << `U`, lots of space wasted

$\longrightarrow$ do not use a direct-access table
$\longrightarrow$ use a hash table

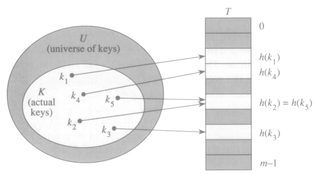
Hash table

- Storage requirement can be reduced to Θ(`k`)
- Searching remains in $\Theta(1)$, however, *in average*

Principle

Direct-access table: stores element with key k in slot k

Hash table: stores element with key k in slot $h(k)$
h, *hashing function*, maps universe into slots
$h : U \rightarrow \{0, 1, \ldots, m-1\}$
An element with key k hashes to slot $h(k)$



Collision:
two keys may hash to the same slot (when h not injective)

Avoiding collision

Make h *appear* to be random: avoids or minimizes collisions

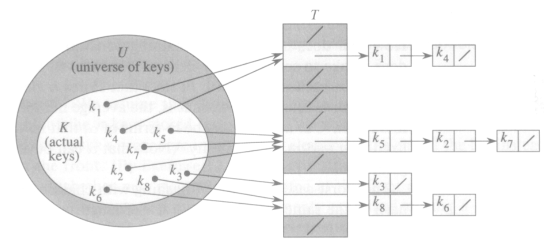
- h must be deterministic: given k, $h(k)$ same
- Since `U` $> m$, no-collisions is impossible

Techniques

- Chaining
- Open addressing

Collision resolution by chaining

Chain elements that hash to the same slot in a linked list



Slot j is a pointer to the head of the list of all elements that hash to j

When $\nexists$ such elements, slot j hashes to `Nil`

Dictionary operations in collision resolution by chaining

`Chained-Hash-Search` (T, k)

search for an element with key k in list $T[h(k)]$

Worst case: proportional to length of list

`Chained-Hash-Insert` (T, x)

insert x at the head of list $T[h(key[x])]$

Worst case: $O(1)$

Exercise 12.2-2

`Chained-Hash-Delete` (T, x)

delete x from list $T[h(key[x])]$

Worst case: $O(1)$ *if list is doubly linked*

Singly-linked list: search for x predecessor to splice x out

Delete and search have same running time

To analyze hashing with chaining, examine search

Analysis of hashing with chaining

Given a hash table T with m slots storing n elements

Load factor for T: average number of elements in a chain

$$\alpha = \frac{n}{m}$$

$\alpha < 1$: on average, less than one element per slot

$\alpha = 1$: on average, one element per slot

$\alpha > 1$: on average, more than one element per slot

Analysis of hashing with chaining

Worst-case:

All n keys hash to same slot, a list of length n

Time for searching: $\Theta(n)$, plus time to compute h

$\rightarrow$ h-table not attractive

Average case:

Depends on how h-function distributes set of keys among m slots, on average

1. Division method
2. Multiplication method
3. Universal hashing

For now, assume **simple uniform hashing**

Simple uniform hashing

Assume:

- Compute $h(k)$ and access slot is in $O(1)$
- Searching element of key k is linear in length of list $T[h(k)]$

Question: What is the number of elements considered by search? i.e., number of elements in list $T[h(k)]$ checked to see if their key is equal to k

1. Search unsuccessful: no element in table has key k
2. Search successful: finds element in table with key k

Result: under assumption of uniform hashing, search is $\Theta(1+\alpha)$ on average

Theorems: 12.1 and 12.2

Theorem 12.1 In a hash table in which collisions are resolved by chaining, an unsuccessful search takes time $\Theta(1+\alpha)$, on the average, under the assumption of simple uniform hashing.

Proof: Simple uniform hashing $\Rightarrow$ any key k is equally likely to hash to any of the m slots

The average time to search unsuccessfully for a key k = average time to search to the end of one of the m lists
Average length of such a list is $\alpha = n/m$
$\Rightarrow$ expected number of elements examined is α
$\Rightarrow$ Total time required $= \Theta(\alpha)$ + time for computing $h(k)$
$\Rightarrow$ Total time required $= \Theta(1+\alpha)$

Theorem 12.2 In a hash table in which collisions are resolved by chaining, a successful search takes time $\Theta(1+\alpha)$, on the average, under the assumption of simple uniform hashing.

Proof
Assumptions:

- Searched key equally likely to be in any of n keys stored
- `Chained-hash-Insert` inserts new element at end of list

Theorem 12.2 ... a successful search takes time $\Theta(1+\alpha)$, on the average ...

Proof (cont')
The expected #elements examined during a successful search is 1 + number of elements examined when sought for element was inserted. Thus, we take the average, over the n items in table, of: 1+ the expected length of the list to which the i element is added
Expected length of list is $(i-1)/m \Rightarrow$ Expected number of elements examined is: $\frac{1}{n}\sum_{i=1}^{n}(1+\frac{i-1}{m})$

$$\frac{1}{n}\sum_{i=1}^{n}(1+\frac{i-1}{m}) = 1+\frac{1}{nm}\sum_{i=1}^{n}(i-1) \quad (1)$$
$$= 1+(\frac{1}{nm})(\frac{(n-1)n}{2}) \quad (2)$$
$$= 1+\frac{\alpha}{2}-\frac{1}{2m} \quad (3)$$

Thus total time required for a successful search is $\Theta(2+\frac{\alpha}{2}-\frac{1}{2m}) = \Theta(1+\alpha)$

Interpretation of Theorem 2.1 and 2.2

If the number of slots, m, is at least proportional to number of elements in table, n, we have $n = O(m) \Rightarrow \alpha = \frac{O(m)}{m} = O(1)$

Thus searching takes constant time on average.

Remember:
Insertion is in $O(1)$, Deletion is in $O(1)$ (doubly-linked lists)

All dictionary operations can be supported in $O(1)$

Hash functions

A good hash function

- can be computed quickly
- satisfies (approx.tely) the simple uniform hashing assumption: each key is equally likely to hash to any of the m slots

Formally:
Assume each key is drawn independently from U with probability distribution P
$P(k)$ is the probability that k is drawn
Simple uniform hashing $\Rightarrow$

$$\sum_{k:h(k)=j} P(k) = \frac{1}{m}, \text{ for } j = 0, 1, \ldots, m-1$$

However, P is usually unknown

Hash functions

1. Division method
2. Multiplication method
3. Universal hashing

Assumption: keys are natural numbers ($\in \mathbb{N}$)

If keys $\notin \mathbb{N}$, find a way to express them as such

Example: strings can be interpreted by interpreting each character as an integer in notation radix-128, using the ASCII character set
Illustration: `pt` can be interpreted as (112.128) +116 = 14452, since p = 112 and t = 116 in the ASCII character set

Division method

$$h(k) = k \mod m$$

Example: $m = 12$ and $k = 100 \Rightarrow h(k) = 4$

- Quick hash function
- Avoid m power of 2
  If $m = 2^p$, then $h(k)$ is just the p lowest order bits of k
  Unless $P(k)$ makes all low-order p-bits patterns equally likely
- Avoid powers of 10 if keys are decimal numbers (as hash function does not depend on all the decimal digits of k)
- Good values for m are primes not too close to exact powers of 2

Example: $n = 2000$ character strings, each character has 8 bits
If $\alpha = 3$ then choose $m = 701$ (prime not too close to a power of 2)

Multiplication method

Two steps:

1. Multiply k by A constant $(0 < A < 1)$, and extract the fractional part of kA
2. Then multiply value but m and take floor of result

$$h(k) = \lfloor m(kA \mod 1) \rfloor$$

where $kA \mod 1 = ka - \lfloor kA \rfloor$

- Value of m is not critical
- We can choose m to make h-function easy to implement

Universal hashing

Worst-case scenario:

- Malicious adversary chooses the keys to be hashed
- Bad choice of hashing, all n keys hash to the same slot

Average retrieval time deteriorates: $\Theta(n)$
Any *fixed* hash function is vulnerable

Way out? Choose a h-function that is random, independent of keys to be stored

The scheme is called **universal hashing**
yields good performance on average
no matter what keys are chosen by adversary

Universal hashing: principal (I)

Select a hash function at random, at run time from a carefully designed class of functions

Randomization guarantees

- that no single input will evoke worst-case behavior
- good average-case performance, no matter what keys are provided as input

Universal hashing: principal (II)

Let $\mathcal{H}$ be a finite set of hash functions that map a universe U of keys into the range $\{0, 1, 2, \ldots, m-1\}$

$\mathcal{H}$ is universal if for every pair of distinct key $x, y, \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(x) = h(y)$ is precisely $|\mathcal{H}|/m$

that is, with a hash function randomly chosen from $\mathcal{H}$, the chance of collision between x and y when $x \neq y$ is exactly $1/m$, which is exactly the chance of a collision if $h(x)$ and $h(y)$ are randomly chosen from the set $\{0, 1, 2, \ldots, m-1\}$