Quicksort

Textbook, Chapter 8

CSCE310: Data Structures and Algorithms

`www.cse.unl.edu/~choueiry/S01-310/`

Berthe Y. Choueiry (Shu-we-ri)
Ferguson Hall, Room 104
`choueiry@cse.unl.edu`, Tel: (402)472-5444

Quicksort

- Worst-case running time is in $\Theta(n^2)$ *(relatively slow)*
- Remarkably efficient: average running time is in $\Theta(n \lg n)$ constants factors hidden in $\Theta(n \lg n)$ quite small
- Sorts in place (no need for external storage)

Quicksort: divide and conquer

```
QUICKSORT(A, p, r)
if p < r
   then q ← PARTITION(A, p, r)
        QUICKSORT(A, p, q)
        QUICKSORT(A, q + 1, r)
```

Consider a subarray $A[p \ldots r]$

Divide: $A[p \ldots r]$ is partitioned into $A[p \ldots q]$ and $A[q+1 \ldots r]$ such that each element in $A[p \ldots q]$ is smaller or equal to each element in $A[q+1 \ldots r]$
Index q is computed here

Conquer: Subarrays $A[p \ldots q]$ and $A[q+1 \ldots r]$ sorted by recursive call to `Quicksort`

Combine: Since subarrays sorted in place, no need to combine them! $A[p \ldots r]$ is sorted

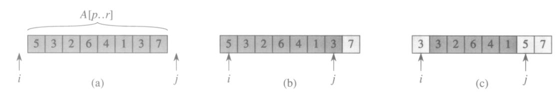
Key to `Quicksort`is the `Partition` procedure

Partition procedure: rearranges $A[p \ldots r]$ in place

Puts elements smaller than x in bottom region of array
Puts elements larger than x in top region

```
PARTITION(A, p, r)
x ← A[p]
i ← p − 1
j ← r + 1
while TRUE
    do repeat j ← j − 1
          until A[j] ≤ x
       repeat i ← i + 1
          until A[i] ≥ x
       if i < j
          then exchange A[i] ↔ A[j]
          else return j
```

Partition procedure: example

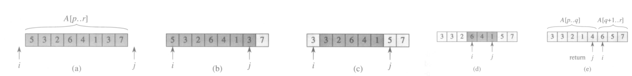


- First, selects $x = A[p]$ as a pivot, around which to partition $A[p \ldots r]$
- Then, grows two regions $A[p \ldots i]$ and $A[j \ldots r]$, from top and bottom of $A[p \ldots r]$ such that

  every element in $A[p \ldots i] \leq x$ and

  every element in $A[j \ldots r] \leq x$
- Initially, the regions are empty ($i = p - 1$ and $j = r + 1$)

Partition procedure: example



- Then, i is incremented and j is decremented until $A[i] \geq x \geq A[j]$
- At this point, $A[i]$ is too big and $A[j]$ is too small to be in respective region

  They should be swapped

  which allows us to continue extending the regions
- This continues until $i \geq j$, with $A[p \ldots r]$ partitioned in $A[p \ldots q]$ and $A[q+1 \ldots r]$, where no element of $A[p \ldots q]$ is larger than no element of $A[q+1 \ldots r]$

Warning: $A[p]$ must be chosen as pivot x

Choose $A[r]$ as pivot, and suppose $A[r]$ is largest element in A

What does `Partition` return?

```
PARTITION(A, p, r)
x ← A[p]
i ← p − 1
j ← r + 1
while TRUE
    do repeat j ← j − 1
         until A[j] ≤ x
       repeat i ← i + 1
         until A[i] ≥ x
       if i < j
         then exchange A[i] ↔ A[j]
         else return j
```

what was it supposed to return?

what happens to `Quicksort`?

Partition: running time

Running time on array $A[p \ldots r]$ is in $\Theta(n)$

```
PARTITION(A, p, r)
x ← A[p]
i ← p − 1
j ← r + 1
while TRUE
    do repeat j ← j − 1
         until A[j] ≤ x
       repeat i ← i + 1
         until A[i] ≥ x
       if i < j
         then exchange A[i] ↔ A[j]
         else return j
```

Exercise 8.1-1

Choice of pivot

Courtesy of C. Cusack

There are various methods that can be used to pick the pivot element, including:

- Use leftmost element as the pivot.
- Use the "median-of-three" rule to pick the pivot.
  - Choose as partitioning: $median(a[p], a[(p+r)/2], a[r])$
  - Partitioning unlikely to generate a "degenerate" partition.
- Use a random element as the pivot
  - yields randomized-partition
  - The average complexity does not depend on the distribution of input sequences

Performance of Quicksort

Is partitioning balanced or unbalanced?
Depends on which elements are used for partitioning

If balanced, `Quicksort` runs asymptotically as fast as `Mergesort`
If unbalanced, `Quicksort` runs asymptotically as slow as `Insertion-sort`

1. Worst-case partitioning
   Runs in $\Theta(n^2)$
2. Best-case partitioning
   Runs in $\Theta(n \lg n)$
3. Average (Balanced) partitioning
   Runs in $\Theta(n \lg n)$

Worst-case partitioning

`Partition` produces 2 regions:
(1) 1 element
(2) $(n-1)$ elements

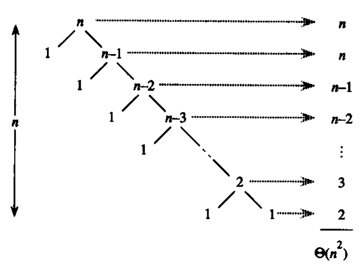
Suppose this happens at every step
(When does this happen?)

But, since partitioning costs $\Theta(n)$ and $T(1) = \Theta(1)$, we have

$$T(n) = T(n-1) + \Theta(n) = \sum_{k=1}^{n} \Theta(k) = \Theta(\sum_{k=1}^{n} k) = \Theta(n^2)$$

Worst-case of `Quicksort` is as bad as that of `Insertion-sort`
And this happens in cases where `Insertion-sort` is linear :—(

Worst-case partitioning: Recurrence tree



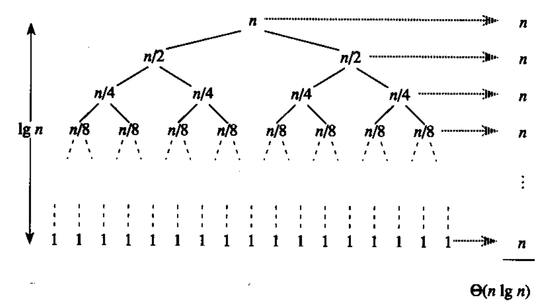
Best-case partitioning

`Partition` produces two regions, each of size $n/2$

Recurrence is: $T(n) = 2T(n/2) + \Theta(n)$

Apply Case 2 of Master Theorem
Solution: $T(n) = \Theta(n \lg n)$

Best-case partitioning: Recurrence tree



Average-case partitioning

Average case much closer to best case than to worst case

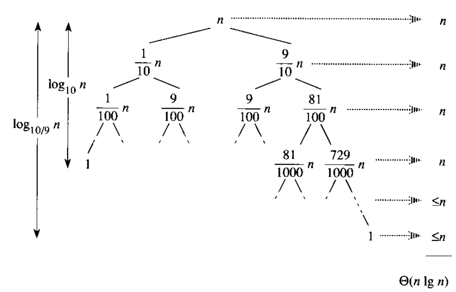
Balanced partitioning: Suppose `Partition` always produces 9-to-1 proportional split

Recurrence is: *replaced* $\Theta(n)$ *by* n

$$T(n) = T(9n/10) + T(n/10) + n$$

Look at recurrence tree..

Proportional split: recurrence tree



Every level has cost n, until reaching boundary condition at depth $\log_1 0n = \Theta(\lg n)$

Then levels have cost at most n, until recursion terminates at depth $\log_{10/9} n = \Theta(\lg n)$

Constant proportionality

Thus, with 9-to-1 proportional split, quicksort runs is in $\Theta(n \lg n)$

It can be shown that the solution is $T(n) = \Theta(n \lg n)$,
whenever the split has constant proportionality

Performance of Quicksort

Courtesy of C. Cusack

Pivot at position i: $T(n) = T(n-i) + T(i-1) + n$
i may be different for each subarray, and at each level of recursion

Best case: $T(n) \leq 2T(n/2) + n$.
Solution: $T(n) = \Theta(n \log n)$.

Worst case:

$$\begin{aligned} T(n) &= T(n-1) + n \\ &= n + (n-1) + \ldots + 1 \\ &= n(n+1)/2 = O(n^2) \end{aligned}$$

Average case:
$T_a(n) = n + \frac{1}{n}\sum_{i=1}^{n}(T_a(i-1) + T_a(n-i))$

Average Case

Courtesy of C. Cusack

- We assume that the pivot has the same probability $(1/n)$ to go into each of the n possible positions. This gives

$$\begin{aligned} T_a(n) &= n + \frac{1}{n}\sum_{i=1}^{n}(T_a(i-1) + T_a(n-i)) \\ &= n + \frac{1}{n}\sum_{i=1}^{n}T_a(i-1) + \frac{1}{n}\sum_{i=1}^{n}T_a(n-i) \\ &= n + \frac{2}{n}\sum_{i=1}^{n}T_a(i-1) \end{aligned}$$

- The last step comes from the fact that the two sums are the same, but in reverse order.
- We have seen that in the best case, $T(n) = O(n \log n)$ and in the worst case, $T(n) = O(n^2)$.
- We guess that $T_a(n) \leq an \log n + b$, for two positive constants a and b.
- We will prove this by induction.

Proof: $T(n) \leq an \log n + b$

Courtesy of C. Cusack

- We can pick a and b so that the condition holds for $T(1)$.
- Assume it holds for all $k < n$. Then

$$\begin{aligned} T_a(n) &= n + \frac{2}{n}\sum_{k=1}^{n}T_a(k-1) \\ &= n + \frac{2}{n}\sum_{k=1}^{n-1}T_a(k) \\ &\leq n + \frac{2}{n}\sum_{k=1}^{n-1}(ak \log k + b) \\ &= n + \frac{2b}{n}(n-1) + \frac{2a}{n}\sum_{k=1}^{n-1}k \log k \end{aligned}$$

Courtesy of C. Cusack

- It can be shown that

$$\sum_{k=1}^{n-1} k \log k \leq \frac{1}{2} n^2 \log n - \frac{1}{8} n^2.$$

- Substituting, we get

$$\begin{aligned}
T_a(n) &= n + \frac{2b}{n}(n-1) + \frac{2a}{n} \sum_{k=1}^{n-1} k \log k \\
&\leq n + \frac{2b}{n}(n-1) + \frac{2a}{n} \left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2 \right) \\
&= n + 2b - \frac{2b}{n} + an \log n - \frac{a}{4} n \\
&= an \log n + b + (n + b - \frac{2b}{n} - \frac{a}{4} n) \\
&\leq an \log n + b + (n + b - \frac{a}{4} n) \\
&\leq an \log n + b.
\end{aligned}$$

- The last step can be obtained by choosing a large enough.
- Thus, $T(n) = O(n \log n)$.