# Graphs

Textbook: Chapter 5, Section 5.4
Figures courtesy of C. Cusack

**CSCE310: Data Structures and Algorithms**
`www.cse.unl.edu/~choueiry/S01-310/`

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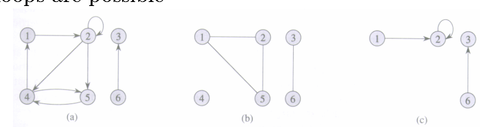
**Graph** is defined by $G(V, E)$

A **graph**: $\begin{cases} \text{V is an unordered finite set of vertices} \\ \text{E is the set of edges} \end{cases}$

→ Self-loops are forbidden: an edge has 2 distinct vertices

A **directed graph** (digraph): $\begin{cases} \text{V is a finite of vertices} \\ \text{E is a binary relation on V} \end{cases}$

→ Self-loops are possible




Vertices are represented as nodes
Edges are represented as arcs
Two vertices between an edge are the **endpoints** of the edge



## Incidence, adjacency

For $(u, v)$ an edge in a **directed** graph $G(V, E)$:

- $(u, v)$ is incident from (or leaves) vertex $u$
- $(u, v)$ is incident to (or enter) vertex $v$
- vertex $v$ is adjacent to vertex $u$, but not vice versa

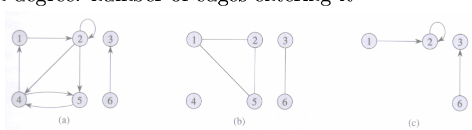
For $(u, v)$ an edge in an **undirected** graph $G(V, E)$:

- $(u, v)$ is incident on vertices $u$ and $v$
- vertex $v$ is adjacent to vertex $u$ and vice versa



## Degree of a vertex

- Indirected graph: number of edges incident on vertex
- Directed graph: out-degree + in-degree
  - Out-degree: number of edges leaving it
  - In-degree: number of edges entering it



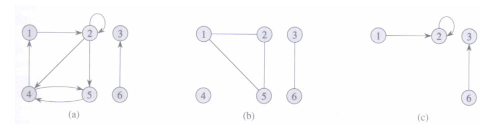
## Path (I)

- A path of length $k$ from a vertex $u_0$ to a vertex $u_k$ in a graph $G(V, E)$ is a sequence $\langle u_0, u_1, u_2, \ldots, u_{k-1}, u_k \rangle$, where $(u_{i-1}, u_i) \in E$ (there is an edge in $G$ incident on each two consecutive vertices)
- Length of path = number of edges in path
- The path contains:
  - vertices $u_0, u_1, u_2, \ldots, u_k$
  - edges $(u_0, u_1), (u_1, u_2), \ldots, (u_{k-1}, u_k)$



## Path (II)

- If $p = \langle u_0, u_1, u_2, \ldots, u_{k-1}, u_k \rangle$, $u_k$ is reachable from $u_0$.
  In a directed graph: $u_0 \overset{p}{\leadsto} u_k$
- Simple path if all vertices are distinct (no crossing)



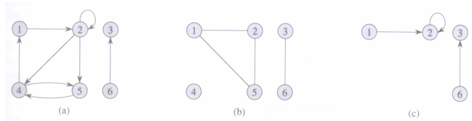

$\langle 1, 2, 5, 4 \rangle$ is a simple path of length 3
$\langle 2, 5, 4, 5 \rangle$ is not a simple path

- A subpath of $p = \langle u_0, u_1, u_2, \ldots, u_{k-1}, u_k \rangle$ is a continuous subsequence of vertices in $p$, $(\langle u_{i \leq 0}, \ldots, u_{j \leq k} \rangle)$



## Cycle in a digraph

- A path $\langle u_0, u_1, u_2, \ldots, u_{k-1}, u_k \rangle$ forms a cycle if $u_0 = u_k$ and $p$ has at least one edge
- A cycle is simple if $u_0, u_1, u_2, \ldots, u_{k-1}, u_k$ are distinct
  Examples: $\langle 1, 2, 4, 1 \rangle$: yes; $\langle 1, 2, 4, 5, 4, 1 \rangle$ no
- A self-loop is a cycle of length ... Example: $\langle 2, 2 \rangle$
- Two path $\langle u_0, u_1, \ldots, u_{k-1}, u_k \rangle$ and $\langle u'_0, u'_1, \ldots, u'_{k-1}, u'_k \rangle$ form the same cycle if $\exists\, j \in \mathbb{N}$ such that
  $u'_i = u_{(i+j) \bmod k}$ for $i = 0, 1, \ldots, k-1$.

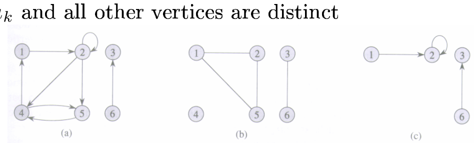



Example: $\langle 1, 2, 4, 1 \rangle$, $\langle 2, 4, 1, 2 \rangle$, $\langle 4, 1, 2, 4 \rangle$,



## Cycle in a undirected graph:

A path $\langle u_0, u_1, u_2, \ldots, u_{k-1}, u_k \rangle$ forms a cycle if $u_0 = u_k$ and all other vertices are distinct

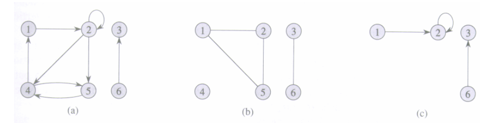



An acyclic graph is a graph with no cycle
A directed graph that is acyclic is denoted **dag**

## Connected graph

Undirected graph:

- An <u>indirected</u> graph is <u>connected</u> if every two vertices are connected by a path
- The connected components form <u>equivalence</u> classes of vertices under the "is reachable from" relation
  Graph (b) has 3 connected components




- An indirected graph is connected if it has exactly one connected component: every vertex is reachable from every other vertex

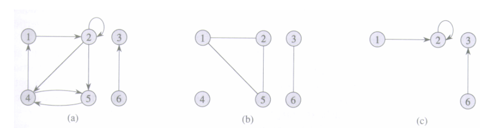
Equivalence relation?



## Connected graph

Directed graph:

- An <u>directed</u> graph is <u>strongly connected</u> if every two vertices are reachable from each other
- The <u>strongly connected components</u> of a graph are equivalence classes of vertices under the "are mutually reachable" relation.
- A directed graph is strongly connected if it has only one strongly connected component

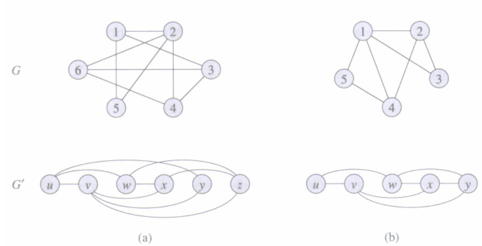



3 strongly connected components: $\{1, 2, 4, 5\}, \{3\}, \{6\}$.



## Isomorphic graphs

Two graphs $G(V, E)$ and $G'(V', E')$ are isomorphic if $\exists$ a bijection $f\colon V \rightarrow V'$ such that $(u, v) \in E$ iff $((f(u), f(v)) \in E'$



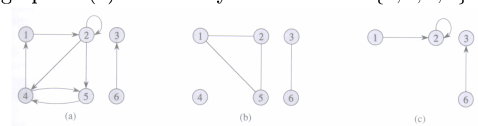



## Subgraph, induced graph

$G'(V', E')$ is a <u>subgraph</u> of $G(V, E)$ if $V' \subseteq V$ and $E' \subseteq E$

Given $V' \subseteq V$, the subgraph of $G$ induced by $V'$ is $G'(V', E')$ where $E' = \{(u, v) \in E \ : u, v \in V'\}$

The subgraph of (a) induced by the vertex set $\{1, 2, 3, 6\}$ is (c):




with $V' = \{(1, 2), (2, 2), (6, 3)\}$

## Directed ⟷ undirected graph

- Undirected graph $G(V, E) \longrightarrow$ directed graph $G'(V, E')$
  Every edge $(u, v)$ is replaced by two directed edges $(u, v)$ and $(v, u)$
  - $V$: same
  - $E'$: $(u, v) \in E' \Leftrightarrow (u, v) \in E$
- Directed graph $G(V, E) \longrightarrow$ undirected graph $G'(V, E')$
  Eliminate self-loops, keep edges, remove directions, remove duplicates
  - $V$: same
  - $E'$: $(u, v) \in E' \Leftrightarrow u \neq v$ and $(u, v) \in E$

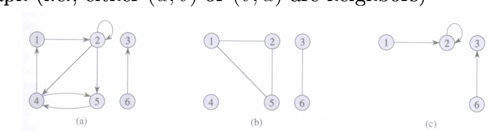
Exercise: 5.4-6



## Neighbor

Two vertices $u$ and $v$ are neighbors

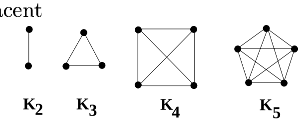
- Undirected graph: $u$ and $v$ are adjacent
- Directed graph: $u$ and $v$ are neighbors in directed version of graph (i.e., either $(u, v)$ or $(v, u)$ are neighbors)



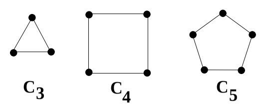



## Special graphs (I)

**Complete graph $K_n$:** undirected graph in which every pair of edge are adjacent



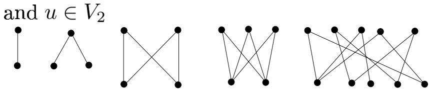

**Cycle graph $C_n$:**






## Special graphs (II)

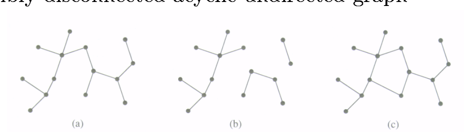
**Bipartite graph $G(V, E)$:** undirected graph in which $V$ is partitioned in $V_1$ and $V_2$: $(u, v) \in E \Rightarrow u \in V_1$ and $v \in V_2$ OR $v \in V_1$ and $u \in V_2$



**DAG:** directed acyclic graph

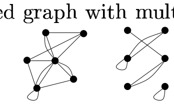
**(Free) tree:** connected acyclic undirected graph

**Forest:** possibly disconnected acyclic undirected graph

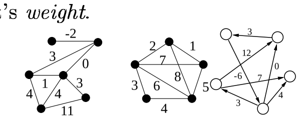


## Special graphs (III)

**Multigraph:** undirected graph with multiple edges and self-loops



**Weighted graph:** a graph (or digraph) with the additional property that each edge $e$ has associated with it a real number $w(e)$ called it's *weight*.




**Hypergraph:** undirected graph with hyperedges, edges connecting an arbitrary number of vertices (instead of only 2)



## Some Theorems: Exercise 5.4-1 *Courtesy of C. Cusack*

- **Theorem 1:** Let $G = (V, E)$ be an undirected graph with $e$ edges.
  Then $2e = \sum_{v \in V} deg(v)$.
- **Proof:** Let $X = \{(e, v) : e \in E, v \in V, e \text{ and } v \text{ are incident}\}$.
  We will compute $\mathtt{X}$ in two ways. Each edge $e \in E$, is incident with exactly 2 vertices. Thus, $\mathtt{X} = 2e$.
  Also, each vertex $v \in V$ is incident with $\mathtt{deg}(v)$ edges.
  Thus, we have that $\mathtt{X} = \sum_{v \in V} \mathtt{deg}(v)$.
  Setting these equal, we have the result ■
- **Corollary 2:** An undirected graph has an even number of vertices of odd degree.



## New terms:

Graph (hypergraph), vertex (vertices), edge (hyperedge), endpoint, directed graph, undirected graph, dag, self-loop, incidence, adjacency, degree of a vertex, out-degree, in-degree, path, path length, subpath, cycle, acyclic, connected graph, connected components, strongly connected (digraph), isomorphic graphs, subgraph, induced graph, neighbor, complete graph, bipartite graph, dag, (free) tree, multigraph, weighted graph,