Energy based approach for task scheduling under time and resource constraints*

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Abstract

This paper presents an energy based approach for the analysis of task scheduling under time and resource constraints. The problem modelling leans on the notion of *time-resource interval* which combines considerations about time and resources. The scheduling of tasks leads to study the interactions between consumer and supplier intervals. A consumer interval (or task) has to be included within a supplier one. The proposed approach aims at refining the temporal bounds of the supplier interval by considering energy consumption by the other consumer intervals. These results may be used to characterize feasible schedules or to detect unfeasibilities.

Key words : task scheduling, time-resource interval, constraints based analysis, temporal reasoning, energy based reasoning.

1 Introduction

This paper deals with the following basic scheduling problem :

- a set of N tasks is to be realized,
- a task is characterized by its duration and has to be achieved within a time window,
- a task needs for its realization a set of resources. It uses a constant known amount of each resource all through its duration,
- each resource is supposed to be always available in a constant amount.

A constraint based analysis [1] of this problem is proposed which aims at characterizing the feasible schedules or at detecting unfeasibilities.
The approach leans on an evaluation of the energy of each resource which is really available for a given task within its time window, by considering the energy consumption of the other tasks. Such an evaluation may be used to point out a lack of energy on a part of the time

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window. This leads to narrow the time window which is really available for the task, through an increasing of earliest starting times or a decreasing of latest finishing times.
This approach is completely different from previous ones [3]. Indeed, the conflicts for using limited resources are not taken into account in a combinatorial way through sequencing conditions, but in a more continuous way through energy consumption which straight acts on time location of the task.
The problem representation is based on the concept of *time-resource interval* [2] which is well suited to the energy based approach and which allows to model more general situations.
The tasks are supposed to be independent, but if not, the updating of limit times can be propagated with no problem through an appropriate potential graph [5][2] which represents the constraints between tasks.

2 Statement of the problem

2.1 Time-resource interval [2]

A *time-resource interval* (later on, we will call it TRI) I is characterized by :

- its temporal bounds C_I (starting time) and F_I (finishing time),
- its resource intensity functions $Q_k^I(t)$ with $t \in [C_I, F_I]$ and $k \in K_I$ where K_I is the resource set associated with I.

A TRI may be defined by its instantaneous characteristics $(C_I, F_I, \{Q_k^I(t)\})$ and/or by its integral ones i.e. by :

- its duration : $D_I = \int_{C_I}^{F_I} dt = F_I - C_I$
- its energy of resource k : $W_k^I = \int_{C_I}^{F_I} Q_k^I(t).dt$

If $\forall k$, $Q_k^I(t)$ is a constant, I is said to be *uniform* [1]. **In the sequel, only this case is studied.**

According to considered problems, TRIs are defined by their instantaneous and/or integral characteristics which are known, unknown, independent or linked - for instance, duration may be expressed as a function of resource intensity.

Two types of TRIs are to be considered :

- *supplier* intervals in which time and resource are allocated,
- *consumer* intervals in which time and resource are required.

A consumer interval is named *task*. Consider a task $i = (C_i, F_i, \{q_k^i\})$, its duration is known and denoted by D_i which is a constant. With each temporal bound, C_i and F_i,

[1] In this case, energy of resource k on interval I may be written : $W_k^I = (F_I - C_I).Q_k^I = D_I.Q_k^I$.

we can associate two real values : $C_i \in [\underline{C}_i, \overline{C}_i]$ and $F_i \in [\underline{F}_i, \overline{F}_i]$. $\underline{C}_i, \overline{C}_i, \underline{F}_i, \overline{F}_i$ respectively represent earliest and latest starting time and earliest and latest finishing time of i. Moreover, supplier intervals are characterized by temporal bounds, known and equal to constants.

2.2 Compulsory energy consumption

Let $\Delta = (C_\Delta, F_\Delta, \{Q_k^\Delta\})$ be a supplier interval and $i = (C_i, F_i, \{q_k^i\})$ [2] a task with duration D_i; the *compulsory energy consumption* [3] of resource k by i on Δ is denoted by $W_k^{i,\Delta}$ and is given by :

$$\begin{aligned} W_k^{i,\Delta} &= max\{0, min[(\underline{C}_i + D_i - C_\Delta), (F_\Delta - (\overline{F}_i - D_i)), F_\Delta - C_\Delta, D_i]\}.q_k^i \\ &= \mathcal{C}^{i,\Delta}.q_k^i \quad \geq 0. \end{aligned}$$

This result is illustrated in Figure 1.

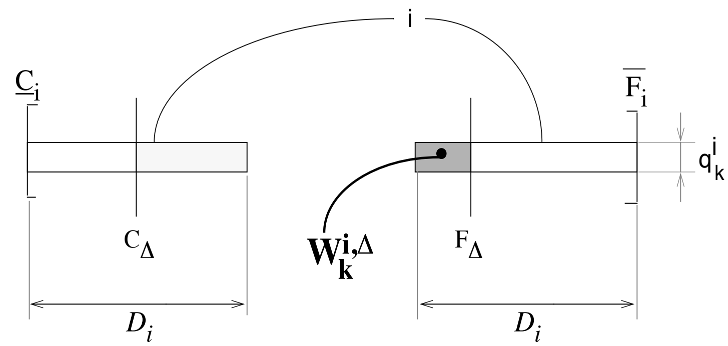

Figure 1. : Compulsory energy consumption by i on Δ.

2.3 Energy based analysis of the interval associated with a task

Consider a task $i = (C_i, F_i, \{q_k^i\})$ with duration D_i and a supplier interval $\delta_C(t) = (\underline{C}_i, t, \{Q_k^{\delta_C(t)}\})$. We define the following variables :

- $W'(t) = (t - \underline{C}_i).Q_k^{\delta_C(t)}$ = energy of resource k possibly provided by $\delta_C(t)$,
- $W''(t) = \sum_{l \neq i} W_k^{l,\delta_C(t)}$ = compulsory energy consumption of resource k by all tasks different from i on $\delta_C(t)$,
- $W_1(t) = W'(t) - W''(t)$ = available energy of resource k on $\delta_C(t)$ for the execution of i,
- $W_2(t, C_i) = min(t - C_i, D_i).q_k^i$ = energy of resource k used by i starting at C_i on $\delta_C(t)$.

[2] Remember that Q_k^Δ and q_k^i are constants.
[3] It appears in a similar form in [4] with the definition of the *compulsory charge*.

Figure 2 shows an example of dynamics of these variables for t varying from $\underline{C}_i$ to $\overline{F}_i$.

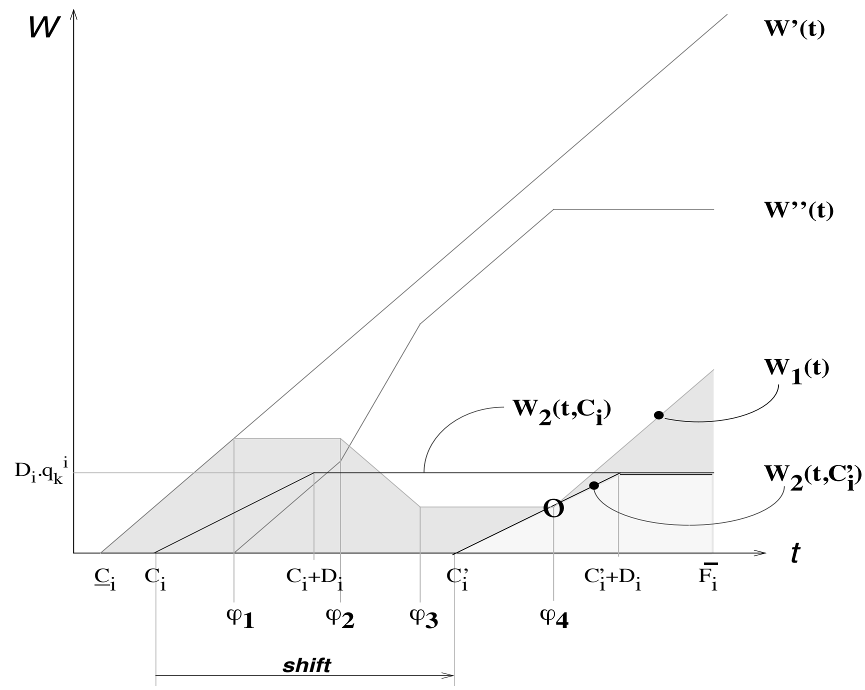

Figure 2.

$\varphi_1, ..., \varphi_4$ are the times associated with the break points of curve $W_1(t)$ [4]. These times are called *remarkable times* of supplier interval $\delta_C(\overline{F}_i)$. By studying curve $W_1(t)$, it is possible to derive some informations about the available energy distribution and its effect on the location in time of i (earliest location in this case). Indeed, $W_1(t) \geq W_2(t, C_i)\ \forall t$ is a necessary condition to start the execution of i at time C_i. This condition may involve a "shift" of curve $W_2(t, C_i)$ to the right - as shown in Figure 2 - (*earliest energy based analysis*); now, i cannot start at $\underline{C}_i$, but only at C_i'. This involves an updating of the earliest starting time of i.

By considering a supplier interval $\delta_F(t) = (t, \overline{F}_i, \{Q_k^{\delta_F(t)}\})$, it is possible to derive symmetrical conclusions. The condition $W_1(t) \geq W_2(F_i, t)$, for t varying from $\overline{F}_i$ to $\underline{C}_i$, may involve an updating of the latest finishing time of i (*latest energy based analysis*). Nevertheless, in this paper, we will only consider supplier intervals such as $\delta_C(t)$. **To simplify notation, we will write $\delta(t)$ instead of $\delta_C(t)$.**

The objective of the analysis is to derive the most accurate effective limit times associated with task i, by considering resources constraints through energy based analysis.

[4] By considering only uniform TRIs, $W'(t)$ is linear and $W''(t)$ is continuous and piece-wise linear. So, we can remark that the break points are the same for $W_1(t)$ and $W''(t)$.

As a result, the goal is to maximize the updating of $\underline{C}_i$. With this in mind, the following approach is proposed :

1. search for all the remarkable times φ_ρ of supplier interval $\delta(\overline{F}_i)$,
2. search for φ_* : the remarkable time which involves the greatest updating of $\underline{C}_i$ and such that $W_1(t) < W_2(t, \underline{C}_i)$, i.e. starting task i at $\underline{C}_i$ is not allowed,
3. if φ_* exists, then update $\underline{C}_i$ to C'_i.

These points are detailed in the following sections.

3 Seeking remarkable times

3.1 Times and associated conditions

Let n be the number of remarkable times of supplier interval $I = (\underline{C}_i, \overline{F}_i, \{Q_k^I\})$ associated with task $i = (C_i, F_i, \{q_k^i\})$, and $\rho = 1, ..., n$ an integer. We note φ_ρ the different values of the remarkable times with $\underline{C}_i < \varphi_\rho < \overline{F}_i$ and $\varphi_\rho < \varphi_{\rho'}$ iff $\rho < \rho'$. The intervals of interest for the earliest energy based analysis are such that :

$$[\underline{C}_i, \varphi_1], [\underline{C}_i, \varphi_2], ..., [\underline{C}_i, \varphi_\rho], ..., [\underline{C}_i, \varphi_n].$$

Within intervals $[\varphi_\mu, \varphi_{\mu+1}]$ $(1 \leq \mu \leq n-1)$, the evolution in time of the available energy for the execution of i is linearly time dependent because TRIs are uniform (cf. Figure 2).

Let $\mathcal{I}_k^I = \{l \ / \ W_k^{l,I} \neq 0\}$ be the set of the tasks which have a compulsory energy consumption of resource k on I, not equal to zero. Consider a task $j = (C_j, F_j, \{q_k^j\})$, with duration D_j such that : $j \neq i$ and $j \in \mathcal{I}_k^I$. Within I, we can associate with j the following remarkable times :

- $\varphi_1(j) = \overline{F}_j - D_j$: latest starting time of j,
- $\varphi_2(j) = \overline{F}_j + \underline{C}_j - \underline{C}_i$: time t when the compulsory energy consumption by j on $(\underline{C}_i, t, \{Q_k^I\})$ is the same for the task wedged either to the right or to the left :
  $[t - (\overline{F}_j - D_j)].q_k^j = (\underline{C}_j + D_j - \underline{C}_i).q_k^j$,
- $\varphi_3(j) = \overline{F}_j$: latest finishing time of j,
- $\varphi_4(j) = \underline{C}_j + D_j$: earliest finishing time of j.

These times are taken into account under a specific condition (over and above that they are included between $\underline{C}_i$ and $\overline{F}_i$ and that $j \in \mathcal{I}_k^I$) given in Table 1.

bound	condition	
$\overline{F}_j - D_j$	—	
$\overline{F}_j + \underline{C}_j - \underline{C}_i$	$\underline{C}_j < \underline{C}_i < \overline{F}_j - D_j$	(E_2)
$\overline{F}_j$	$\underline{C}_i \leq \underline{C}_j < \overline{F}_j < \overline{F}_i$	(E_3)
$\underline{C}_j + D_j$	$\overline{F}_j - D_j \leq \underline{C}_i$	(E_4)

Table 1

Proof

(E_2) 1. to be able to wedge j to the left, it is necessary to have : $\underline{C}_j < \underline{C}_i$.

2. furthermore, to be able to wedge j to the right and to take into account this situation, it is necessary to have : $\overline{F}_j - D_j > \underline{C}_i$. Indeed, if $\overline{F}_j - D_j \leq \underline{C}_i$, then the consumption of j wedged to the right is necessary greater than or equal to the consumption of j wedged to the left.

(E_3) is evident.

(E_4) if $\overline{F}_j - D_j \leq \underline{C}_i$ the consumption of j wedged to the left is necessary less than or equal to the consumption of j wedged to the right.

Figure 3 shows the situations squaring with the conditions (E_2, E_3, E_4).

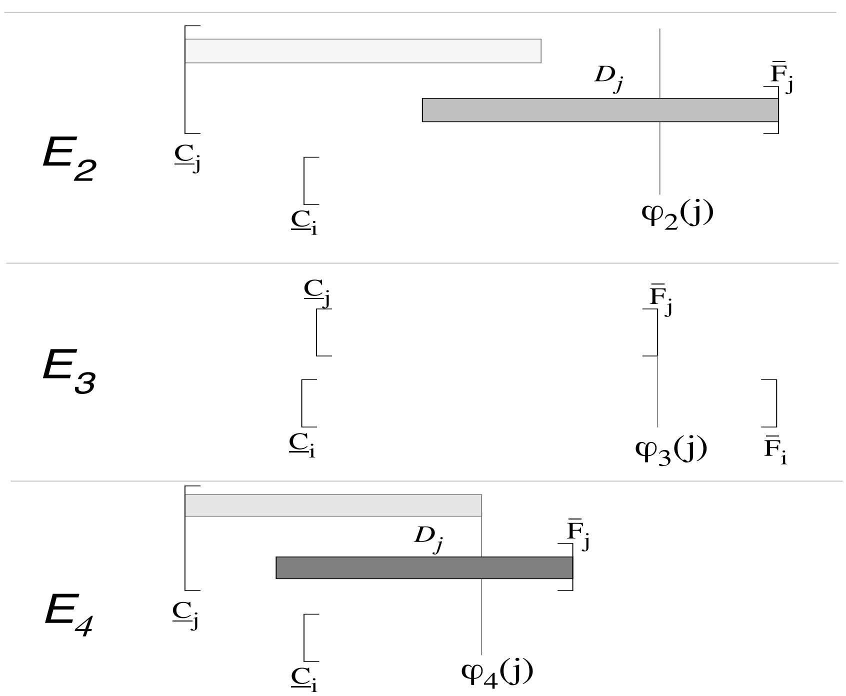

Figure 3.

Remark 1 : the exclusive character between some conditions on the bounds enables to determine, at the most, *two* remarkable times of

I by task :

$$\varphi_1 \text{ or } \varphi_2 \text{ or } \varphi_3 \text{ or } \varphi_4 \text{ or } (\varphi_1 \text{ and } \varphi_2) \text{ or } (\varphi_1 \text{ and } \varphi_3)$$

Moreover, $\underline{F}_i$ and $\overline{F}_i$ are always remarkable times to be considered; so, $\varphi_n = \overline{F}_i$.

3.2 Choice of φ_*

Let $i = (C_i, F_i, \{q_k^i\})$ be a task and $\delta(\varphi_\rho) = (\underline{C}_i, \varphi_\rho, \{Q_k^{\delta(\varphi_\rho)}\})$ a supplier interval; we saw (cf. **2.3**) that the execution of i is possible only if $W_1(t) \geq W_2(t, C_i)$, $\forall t$. Among the whole remarkable times, we search for φ_* which induces the greatest updating of $\underline{C}_i$ and such that $W_1(\varphi_*) < W_2(\varphi_*, \underline{C}_i)$, i.e. :

$$(\varphi_* - \underline{C}_i).Q_k^{\delta(\varphi_*)} - \sum_{l \neq i} W_k^{l,\delta(\varphi_*)} < min(\varphi_* - \underline{C}_i, D_i).q_k^i$$

Remark 2 : in order to minimize the number of remarkable times φ_ρ to be tested, they are examined according to decreasing values of ρ. Indeed, for many examples, the greatest updating of $\underline{C}_i$ is reached for $\varphi_* = \max_\rho \varphi_\rho$.

4 Updating of $\underline{C}_i$

By considering same task i and supplier interval $\delta(\varphi_*) = (\underline{C}_i, \varphi_*, \{Q_k^{\delta(\varphi_*)}\})$, the following updating rule may be derived :

$$\begin{cases} \text{if} \quad (\varphi_* - \underline{C}_i).Q_k^{\delta(\varphi_*)} < \sum_{l \neq i} W_k^{l,\delta(\varphi_*)} + min(\varphi_* - \underline{C}_i, D_i).q_k^i \\ \\ \text{then} \quad [F_i - max(\varphi_*, \underline{C}_i + D_i)].q_k^i \geq S_k^{\delta(\varphi_*)} \quad \text{i.e.} \quad F_i \geq \dfrac{S_k^{\delta(\varphi_*)}}{q_k^i} + max(\varphi_*, \underline{C}_i + D_i) \end{cases}$$

where

$$S_k^{\delta(\varphi_*)} = \sum_{l \neq i} W_k^{l,\delta(\varphi_*)} + min(\varphi_* - \underline{C}_i, D_i).q_k^i - (\varphi_* - \underline{C}_i).Q_k^{\delta(\varphi_*)}.$$

$S_k^{\delta(\varphi_*)}$ stands for the minimal energy of resource k required by i which has to be located out of $\delta(\varphi_*)$. Applying this rule may involve a straight updating of $\underline{F}_i$. This updating leads to a new definition of intervals δ, and the process may be iterated. So, the evolution of the updated value of $\underline{F}_i$ may be expressed by the recurrent series defined by :

$$\underline{F}_i^0 = \underline{F}_i \quad ; \quad \underline{F}_i^{p+1} = \frac{S_k^{\delta^p(\varphi_*^p)}}{q_k^i} + max(\varphi_*^p, \underline{C}_i^p + D_i)$$

with $\delta^p(\varphi_*^p) = (\underline{C}_i^p, \varphi_*^p, \{Q_k^{\delta^p(\varphi_*^p)}\})$.

Since $\underline{C}_i = \underline{F}_i - D_i$, putting $\varphi_*^p - min(\varphi_*^p - \underline{C}_i^p, D_i)$ in place of $max(\varphi_*^p, \underline{C}_i^p + D_i) - D_i$, yields :

$$\underline{C}_i^{p+1} = \frac{\sum_{l \neq i} W_k^{l,\delta^p(\varphi_*^p)} - (\varphi_*^p - \underline{C}_i^p).Q_k^{\delta^p(\varphi_*^p)}}{q_k^i} + \varphi_*^p$$

Putting $W_k^{l,\delta^p(\varphi_*^p)} = \mathcal{C}^{l,\delta^p(\varphi_*^p)}.q_k^l$ (cf. **2.2**)

and $\begin{cases} \alpha_k^{i,\delta^p(\varphi_*^p)} = \dfrac{Q_k^{\delta^p(\varphi_*^p)}}{q_k^i} \geq 1 \\ \\ \beta_k^{i,l} = \dfrac{q_k^l}{q_k^i} > 0 \end{cases}$

we get :

$$\boxed{\underline{C}_i^{p+1} = \underline{C}_i^p.\alpha_k^{i,\delta^p(\varphi_*^p)} + \varphi_*^p(1 - \alpha_k^{i,\delta^p(\varphi_*^p)}) + \sum_{l \neq i} \mathcal{C}^{l,\delta^p(\varphi_*^p)}.\beta_k^{i,l}}$$

`Particular case` : for disjunctive resources constraints (all the q and Q are equal to the same value), $\alpha_k^{i,\delta^p(\varphi_*^p)}$ and $\beta_k^{i,l}$ are equal to 1, so we can write :

$$\underline{C}_i^{p+1}(disjunctive) = \underline{C}_i^p + \sum_{l \neq i} \mathcal{C}^{l,\delta^p(\varphi_*^p)}.$$

Series $\underline{C}_i^{p+1}$ converges to $\underline{C}_i^*$ in a finite or infinite number of iterations (see Examples).

5 Examples

5.1 Example 1 : disjunctive problem

$$\underline{\textbf{Reminder}} : \underline{C}_i^{p+1} = \underline{C}_i^p + \sum_{l \neq i} \mathcal{C}^{l,\delta^p(\varphi_*^p)}.$$

Consider five tasks a, b, c, d, e that share a common unique resource. The task characteristics are given in Table 2.

i	a	b	c	d	e
$\underline{C}_i$	2	7	1	5	1
$\overline{F}_i$	8	14	9	13	14
D_i	2	1	5	2	2

Table 2

`Earliest energy based analysis according to task` e

The supplier intervals to be considered are such as : $[1, \varphi]$ with $1 < \varphi \leq 14$.
The dynamics of the different energies for e is displayed in Figure 4.

The remarkable time φ_* is given by the greatest value of φ such that $W_1 < W_2$.
Following on `Remark 2`, we get Table 3.

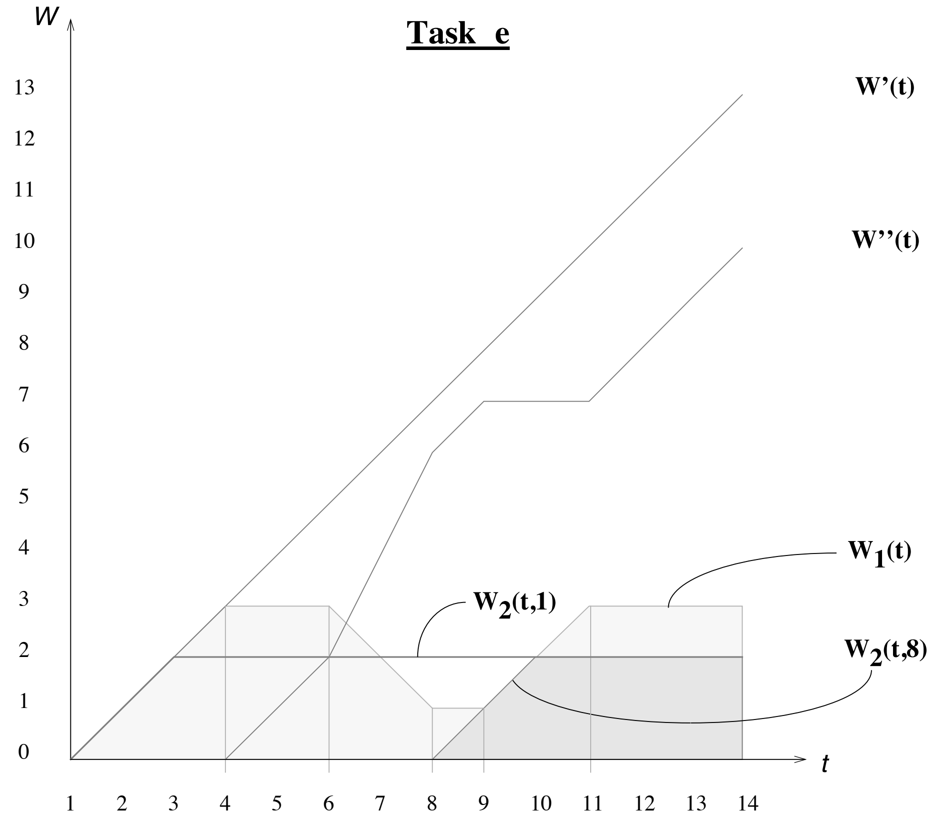

Figure 4.

φ	14	13	11	9	8	6	4	3
W_1	3	3	3	1	—	—	—	—
W_2	2	2	2	2	—	—	—	—
$W_1 < W_2$	NO	NO	NO	YES	—	—	—	—

Table 3

We have : $\varphi_* = 9$ and $\delta(\varphi_*) = [1, 9]$.

$$\sum_{l \neq e} \mathcal{C}^{l,\delta(\varphi_*)} = \mathcal{C}^{a,\delta(\varphi_*)} + \mathcal{C}^{c,\delta(\varphi_*)} = 7.$$

Hence : $\boxed{\underline{C_e^*} = 1 + 7 = 8}$

Earliest energy based analysis according to task d

For d, $5 < \varphi \leq 13$ (see Figure 5 and Table 4).

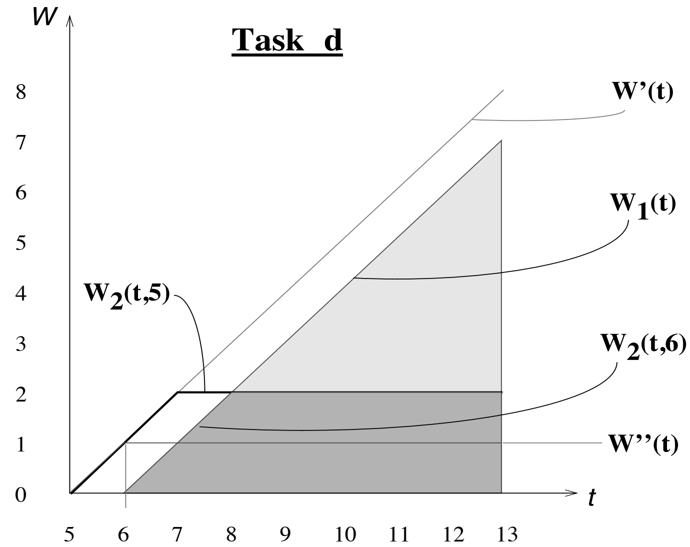

Figure 5.

φ	13	7	6
W_1	7	1	0
W_2	2	2	1
$W_1 < W_2$	NO	NO	YES

Table 4

$$\varphi_* = 7 \text{ and } \delta(\varphi_*) = [5,7].$$

$$\sum_{l \neq d} \mathcal{C}^{l,\delta(\varphi_*)} = \mathcal{C}^{c,\delta(\varphi_*)} = 1.$$

$$\text{Hence : } \boxed{\underline{C}_d^* = 5 + 1 = 6}$$

Table 5 shows the final result of the earliest and the latest energy based analysis.

i	a	b	c	d	e
$\underline{C}_i$	2	7	1	6	8
$\overline{F}_i$	8	14	9	13	14
D_i	2	1	5	2	2

Table 5

5.2 Example 2 : cumulative problem

5.2.1 Case without iteration

Consider now three tasks using one resource available in a quantity greater than one, and with the following characteristics :

i	$\underline{C}_i$	$\overline{F}_i$	D_i	q_k^i
a	0	5	4	1
b	0	3	2	1
c	0	5	3	1

Q_k^I
2

Table 6

`Earliest energy based analysis according to task` c

The intervals of interest satisfy : $[0, \varphi]$ with $0 < \varphi \leq 5$ (see Figure 6 and Table 7).

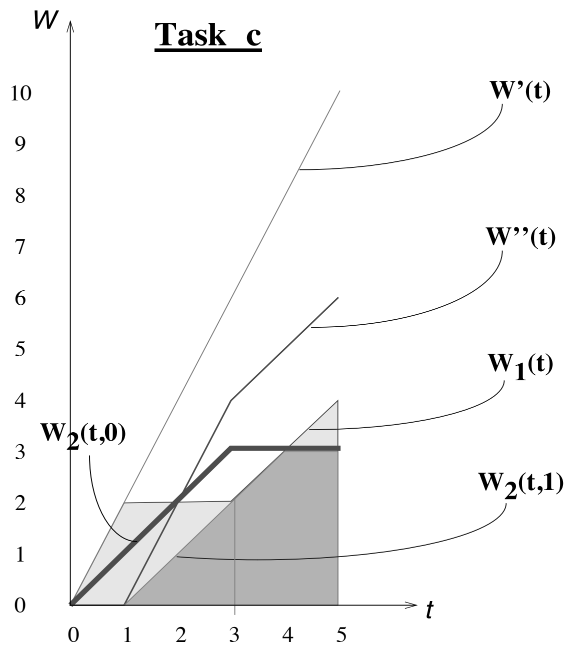

Figure 6.

φ	5	3	1
W_1	4	2	—
W_2	3	3	—
$W_1 < W_2$	NO	YES	—

Table 7

$$\varphi_* = 3 \text{ and } \delta(\varphi_*) = [0, 3].$$

$$\begin{aligned}\sum_{l \neq c} \mathcal{C}^{l,\delta(\varphi_*)}.\beta_k^{c,l} &= \mathcal{C}^{a,\delta(\varphi_*)}.\beta_k^{c,a} + \mathcal{C}^{b,\delta(\varphi_*)}.\beta_k^{c,b} \\ &= 2 \times 1 + 2 \times 1 = 4 \\ \alpha_k^{c,\delta(\varphi_*)} &= 2.\end{aligned}$$

Hence : $\boxed{\underline{C}_c^* = 0 + 3 \times (1 - 2) + 4 = 1}$

The final results are given below :

i	$\underline{C}_i$	$\overline{F}_i$	D_i	q_k^i
a	0	5	4	1
b	0	2	2	1
c	2	5	3	1

Q_k^I
2

Table 8

5.2.2 Case of several iterations

This very elementary example (Table 9) shows an updating process which converges with an infinite number of iterations.

i	$\underline{C}_i$	$\overline{F}_i$	D_i	q_k^i
a	0	1	1	1
b	0	3	2	2

Q_k^I
2

Table 9

`Earliest energy based analysis according to task` b

Iteration 1
For b, we have : $0 < \varphi \leq 3$. We get Figure 7 and Table 10.

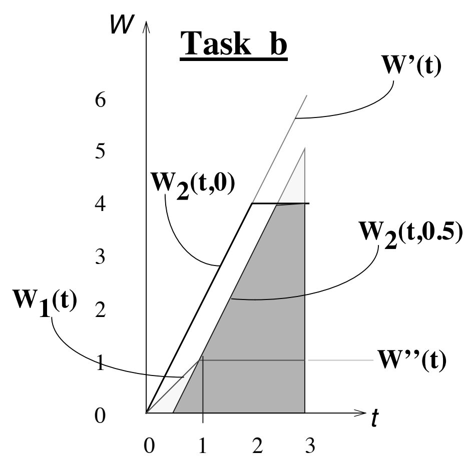

Figure 7.

φ	3	2	1
W_1	5	3	1
W_2	4	4	2
$W_1 < W_2$	NO	NO	YES

Table 10

$$\varphi_*^1 = 2 \text{ and } \delta(\varphi_*^1) = [0, 2].$$

$$\sum_{l \neq b} \mathcal{C}^{l,\delta(\varphi_*^1)}.\beta_k^{b,l} = \mathcal{C}^{a,\delta(\varphi_*^1)}.\beta_k^{b,a} = 0.5$$

$$\alpha_k^{b,\delta(\varphi_*^1)} = 1.$$

$$\text{Hence : } \underline{C}_b^1 = 0 + 0 + 0.5 = 0.5$$

Iteration 2

We have the new problem :

i	$\underline{C}_i$	$\overline{F}_i$	D_i	q_k^i
a	0	1	1	1
b	0.5	3	2	2

Q_k^I
2

Table 11

The evolution of energies are for : $0.5 < \varphi \leq 3$ (see Figure 8).
We have $\varphi_*^2 = 2.5$ and thus, $\delta(\varphi_*^2) = [0.5, 2.5]$.

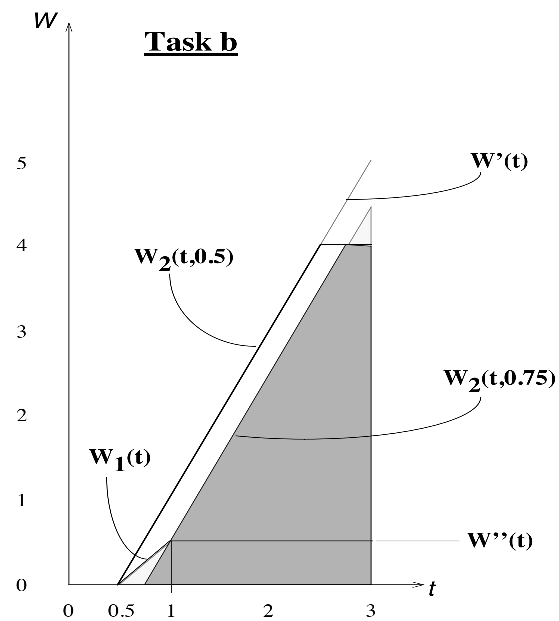

Figure 8.

$$\sum_{l \neq b} \mathcal{C}^{l,\delta(\varphi_*^2)}.\beta_k^{b,l} = \mathcal{C}^{a,\delta(\varphi_*^2)}.\beta_k^{b,a} = 0.25$$

$$\text{Hence : } \underline{C}_b^2 = 0.5 + 0 + 0.25 = 0.75$$

$\vdots$

And the procedure is iterated until : $\sum_{l \neq b} C^{l,\delta(\varphi_*^*)}.\beta_k^{b,l} = 0.$

$$\text{Then : } \boxed{\underline{C}_b^* = 1}$$

Thus, we obtain :

i	$\underline{C}_i$	$\overline{F}_i$	D_i	q_k^i
a	0	1	1	1
b	1	3	2	2

Q_k^I
2

Table 12

6 Conclusion

The results which have been presented above use the energy concept to analyse the interactions between time and resources constraints in task scheduling. The energy based approach seems to be promising for it allows to deal with disjunctive and cumulative resources constraints in an homogeneous way, and for it lends itself to various extensions (non uniform intervals, discontinuous time windows, resource dependent durations...).

At the present time, there exists two modules implemented in Prolog : the first one [3] is based on sequencing conditions and the second one was developped on the basis of the results presented herein.

Comparing the energy based approach and the one which is based on sequencing conditions should be carried out according to the complexity and the completeness of feasibility characterization.

References

[1] **J. ERSCHLER**
Analyse sous contraintes et aide à la décision pour certains problèmes d'ordonnancement
Thèse de Doctorat d'état, LAAS, Université Paul Sabatier, Toulouse, 1976.

[2] **J. ERSCHLER, P. LOPEZ, C. THURIOT**
Raisonnement temporel sous contraintes de ressources et problèmes d'ordonnancement
Revue d'Intelligence Artificielle, 1990 (to appear).

[3] **P. ESQUIROL**
Règles et processus d'inférence pour l'aide à l'ordonnancement de tâches en présence de contraintes
Thèse de Doctorat de l'Université Paul Sabatier, LAAS, Toulouse, 1987.

[4] **A. LAHRICHI**
Ordonnancements : les notions de bosse, de partie obligatoire et leurs applications aux problèmes cumulatifs
C.R. Acad. Sc. Paris, t. 294, série I–16, pp. 209–211, 1982.

[5] **B. ROY**
Algèbre moderne et théorie des graphes, tome 2, chap. VIII
Dunod, Paris, 1970.