Week 16 Recitation

Robert Woodward

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- Questions about lecture / homework so far?
- Find the solution to the recurrence relation:

$$\begin{aligned} a_n &= 2a_{n-1} + 8a_{n-2} \\ a_0 &= 3 \\ a_1 &= 4 \end{aligned}$$

This relation is a linear homogeneous recurrence relation of degree 2 because:

  - The right-hand side has only multiples of previous terms of the sequence and coefficients are all constants. Therefore, it is linear.
  - No terms occur that are not multiples of a_j (i.e., no non-recursive cost). Therefore, it is homogeneous.
  - It is expressed in terms of the $(n-2)^{th}$ term of the sequence. Therefore, it is of degree 2).

We know that a solution to solve this recurrence relation is of the form $a_n = r^n$ where r is some real constant. Replacing the solution in the recurrence relation, we get:

$$r^n = 2r^{n-1} + 8r^{n-2}.$$

Dividing by r^{n-2}, we get:

$$r^2 = 2r^1 + 8.$$

Thus, the *characteristic equation* of this recurrence relation is:

$$r^2 - 2r - 8 = (r+2)(r-4) = 0.$$

This characteristic equation has the roots $r_1 = -2$ and $r_2 = 4$; Therefore, the solution of the recurrence relation is

$$a_n = \alpha_1(-2)^n + \alpha_2 4^n.$$

Plugging in our initial conditions we get

$$\begin{aligned} 3 &= \alpha_1 + \alpha_2 \\ 4 &= -2\alpha_1 + 4\alpha_2 \end{aligned}$$

Solving for $\alpha_1 = 3 - \alpha_2$, we get $4 = -2(3-\alpha_2) + 4\alpha_2 \Rightarrow 4 = -6 + 2\alpha_2 + 4\alpha_2 \Rightarrow \frac{5}{3} = \alpha_2$. Therefore, $\alpha_1 = \frac{4}{3}$ and $\alpha_2 = \frac{5}{3}$.

Putting the values of α_1, α_2 back in the solution form, we obtain the following solution of the recurrence relation given the boundary conditions

$$a_n = \frac{4}{3}(-2)^n + \frac{5}{3}4^n.$$

- Solve the following linear non-homogeneous recurrence relation:

$$\begin{aligned} a_n &= 2a_{n-1} - 8a_{n-2} + n && (1) \\ a_0 &= 3 && (2) \\ a_1 &= 4 && (3) \end{aligned}$$

We notice that $f(n)$ is polynomial n. We will solve this problem using Theorem 6 on page 469, which covers this case, the case that $f(n)$ is an exponential in n, and the case where $f(n)$ is a product of a polynomial and an exponential in n.

First, we solve the associated linear *homogeneous* recurrence relation, which happens to be the one above :-). Its solution is:

$$a_n = \alpha_1(-2)^n + \alpha_2 4^n.$$

Next, we find a particular solution for the given non-homogeneous term. Theorem 6 applies to $f(n)$ of the form:

$$f(n) = (b_t n^t + b_{t-1}n^{t-1} + \ldots + b_1 n + b_0)s^n.$$

In our case, $f(n) = n$ and s is 1. Since our s is *not* a root of our characteristic equation (*relief*), there is a particular solution of the form:

$$a^p = (p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)s^n.$$

For us, the particular solution is $a^p = p_1 n + p_0$. Plugging the particular solution in Equation (1), we get:

$$\begin{aligned} a^p = p_1 n + p_0 &= 2(p_1(n-1) + p_0) - 8(p_1(n-2) + p_0) + n \\ &= 2p_1 n - 2p_1 + 2p_0 - 8p_1 n + 16p_1 - 8p_0 + n \end{aligned}$$

Moving all terms to one side of the equation, we get:

$$(7p_1 - 1)n + (-14p_1 + 7p_0) = 0.$$

Given that $n \neq 0$, we must have the following:

$$\begin{aligned} 7p_1 - 1 &= 0 \\ -14p_1 + 7p_0 &= 0 \end{aligned}$$

Now, $7p_1 - 1 = 0 \Rightarrow 7p_1 = 1 \Rightarrow p_1 = \frac{1}{7}$.
Further, $-14p_1 + 7p_0 = 0 \Rightarrow -14(\frac{1}{7}) + 7p_0 = 0 \Rightarrow -2 + 7p_0 = 0 \Rightarrow 7p_0 = 2 \Rightarrow p_0 = \frac{2}{7}$.
We have thus found the particular solution: have $a^p = \frac{1}{7}n + \frac{2}{7}$. Therefore,

$$\begin{aligned} a_n &= a^h + a^p \\ a_n &= (\alpha_1(-2)^n + \alpha_2 4^n) + (\frac{1}{7}n + \frac{2}{7}). \end{aligned}$$

To determine the values of α_1, α_2, we plug in our initial conditions:

$$\begin{aligned} 3 &= \frac{2}{7} + \alpha_1 + \alpha_2 \\ 4 &= \frac{1}{7} + \frac{2}{7} + -2\alpha_1 + 4\alpha_2 \end{aligned}$$

Or:

$$\begin{aligned} \frac{19}{7} &= \alpha_1 + \alpha_2 \\ \frac{25}{7} &= -2\alpha_1 + 4\alpha_2 \end{aligned}$$

Solving for $\alpha_1 = \frac{19}{7} - \alpha_2$, we get $\frac{25}{7} = -2(\frac{19}{7} - \alpha_2) + 4\alpha_2 \Rightarrow \frac{25}{7} = -\frac{38}{7} + 2\alpha_2 + 4\alpha_2 \Rightarrow \frac{63}{7} = 6\alpha_2 \Rightarrow \frac{63}{42} = \alpha_2$ Replacing α_2 by its value, we get $\alpha_1 = \frac{-13}{42}$.
Replacing α_1, α_2, we get

$$a_n = \frac{1}{7}n + \frac{2}{7} + \frac{63}{42}(-2)^n + \frac{-13}{42}4^n.$$

- Give the asymptotic characterization for $T(n) = 3T(n/4) + 8n^3$.

  Remember Master Theorem, when we have $T(n) = aT(n/b) + f(n)$ where:

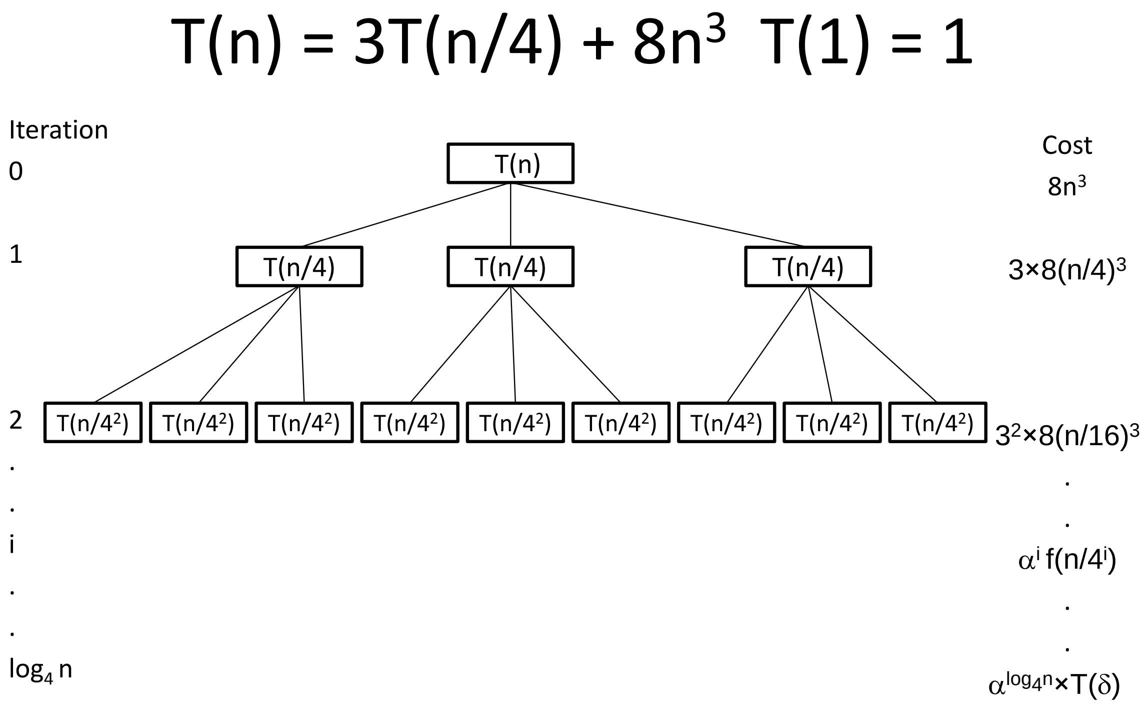
  - $T(n)$ is monotone
  - $f(n) \in \Theta(n^d)$ where $d \geq 0$,
  - b is a constant

we can use it to classify our recurrence relation as follows:

$$T(n) \text{ is } \begin{cases} \mathcal{O}(n^d) & a < b^d \\ \mathcal{O}(n^d \log n) & a = b^d \\ \mathcal{O}(n^{\log_b a}) & a > b^d \end{cases}$$

Therefore, we can use Master's theorem and $a = 3$, $b = 4$ and $d = 3$. Therefore, $T(n)$ is $\mathcal{O}(n)$ because $a < b^d (3 < 64)$.

- Draw the recurrence tree for $T(n) = 3T(n/4) + 8n^3$, $T(1) = 1$.



- No Quiz