Week 15 Recitation

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• Questions about lecture / homework so far?

• Rosen 2.4, Problem 17(a), Page 162: Compute $\sum_{i=1}^{2} \sum_{j=1}^{3} (i + j)$

\[
\begin{align*}
\sum_{i=1}^{2} \sum_{j=1}^{3} (i + j) &= \sum_{i=1}^{2} \left( \sum_{j=1}^{3} i + \sum_{j=1}^{3} j \right) \quad (1) \\
&= \sum_{i=1}^{2} \left( i \sum_{j=1}^{3} 1 + \sum_{j=1}^{3} j \right) \quad (2) \\
&= \sum_{i=1}^{2} \left( i \times (3 - 1 + 1) + \frac{3(3 + 1)}{2} \right) \quad (3) \\
&= \sum_{i=1}^{2} (3i + 6) \quad (4) \\
&= \sum_{i=1}^{2} i + 6 \sum_{i=1}^{2} 1 \quad (5) \\
&= 3 \times \frac{2(2 + 1)}{2} + 6(2 + 1 - 1) \quad (6) \\
&= 3 \times 3 + 6 \times 2 \quad (7) \\
&= 21 \quad (8)
\end{align*}
\]

- In (1), we split the terms in the inner summation.
- In (2), we move $i$ outside the second summation because the index of the summation does not depend on $i$. It is as if $i$ was a constant.
- In (3), we compute the two inner summations.
- In (5), we split the terms in the unique summation left.
- The rest should be straightforward.

• Show how to do summation in Maple
1. SSH into cse.unl.edu
2. Type in “maple” to launch Maple
3. Type in “sum(What to Sum, Over what range);”. For example, to sum $\sum_{k=1}^{10} k^2$, you would type “sum(k^2, k = 1..10);”. Another example, to sum $\sum_{k=1}^{n} k^2$, you would type “sum(k^2,k = 1..n);”.

• Rosen 4.1, Problem 5, Page 280.

Using the Principle of Mathematical Induction, we will prove that

$$\forall n \in \mathbb{N}, 1^2 + 3^2 + 5^2 + \ldots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3.$$ 

We complete the following four steps:

1. **State the propositional property:**

   $$P(n) : 1^2 + 3^2 + 5^2 + \ldots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3.$$ 

2. **Basic Step:** We verify that $P(1)$ is true.

   We must form the basic step. First, we can write out what $P(1)$ is.

   $$P(1) : 1^2 = (1 + 1)(2 \times 1 + 1)(2 \times 1 + 3)/3 = 2 * 3 * 5 \Rightarrow 10 = 10.$$ 

   Second, we need to verify that this holds: $1 + 9 = (2)(2 + 1)(2 + 3)/3 \Rightarrow 10 = 2 * 3 * 5 \Rightarrow 10 = 10.$

3. **State the Inductive Hypothesis:** We assume that $P(k)$ holds for all $k$ positive numbers:

   $$\forall k \in \mathbb{N}, 1^2 + 3^2 + 5^2 + \ldots + (2k + 1)^2 = (k + 1)(2k + 1)(2k + 3)/3.$$ 

4. **Inductive Step:** We show that the conditional statement $P(k) \Rightarrow P(k + 1)$ is true for all positive integers $k$. That is, we prove that $P(k + 1)$ must hold assuming that $P(k)$ holds for all positive integers $k$. Consider the statement $P(k)$:

   $$1^2 + 3^2 + \ldots + (2k + 1)^2 = (k + 1)(2k + 1)(2k + 3)/3.$$ 

   Adding the term $(2k + 3)^2$ on both sides of the equality, we get:

   
   \[
   1^2 + 3^2 + \ldots + (2k + 1)^2 + (2k + 3)^2 = (k + 1)(2k + 1)(2k + 3)/3 + (2k + 3)^2 \\
   = (2k + 3)[(k + 1)(2k + 1)/3 + (2k + 3)] \\
   = (2k + 3)[(2k^2 + 3k + 1)/3 + (6k + 9)/3] \\
   = (2k + 3)[(2k^2 + 9k + 10)/3] \\
   = (2k + 3)(2k + 5)(k + 2)/3 \\
   = ((k + 1) + 1)(2[k + 1] + 1)(2[k + 1] + 3)/3
   \]

   Therefore, $P(k+1)$ holds, so we have shown that $P(k) \Rightarrow P(k+1)$. In conclusion, the statement holds by the Principle of Mathematical Induction. □

Using the Second Principle of Mathematical Induction (or Strong Induction), we will prove that:

Using just 3-cent stamps and 5-cent stamps, we can form \( n \) cents.

We complete the following four steps:

1. **State the propositional property:** \( P(n) : n \) cents can be formed using just 3-cent stamps and 5-cent stamps for \( n \geq 8 \).
2. **Basic Step:** We verify that \( P(8) \) holds. \( P(8) : 1 \) 5-cent stamp and 1 3-cent stamp.
3. **State the Inductive Hypothesis:** We assume that \( P(j) \) holds for all \( 8 \leq j \leq k \).
4. **Inductive Step:** We show that the conditional statement \([P(1) \land P(2) \land \ldots \land P(k)] \rightarrow P(k+1)\).

By the inductive hypothesis, \( P(k - 2) \) must hold. Therefore, \( P(k+1) \) must hold because it requires the same amount of coins as \( P(k - 2) \), plus 1 more 3-cent stamp. Or, \( P(k+1) : P(k - 2) \) and one more 3-cent stamp. Therefore, \( P(k+1) \) holds, so we have shown that \([P(1) \land P(2) \land \ldots \land P(k)] \rightarrow P(k+1)\). In conclusion, the statement holds by the Second Principle of Mathematical Induction. □

(Last 15 minutes) Quiz