Symmetries in CSP

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April 18, 2009
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Historical Note
What is Symmetry?

Symmetry

- Defined as “patterned self-similarity”.
- Generated by a transformation $S$ of an object $O_1$ into $O_2$.
- $S(O_1)$ is not distinguishable from $O_2$.
- Common $S$ are translation, rotation and reflection.
Crafting a Paper Snowflake

How to cut out a snowflake from a piece of paper?
Crafting a Paper Snowflake

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Crafting a Paper Snowflake

How to cut out a snowflake from a piece of paper?
In general biological science problems have many geometric symmetries.
Why is Symmetry?

- CSP = (V, D, C) ∈ NPC, but ∃ islands of tractability.
- Using the structure of CSP to reduce complexity, or to reduce the problem size.
- Symmetry can occur in V, D and C ex. ALL-DIFF constraint.
- CSP’s elements that are symmetric under S create an equivalence class.
- Property detected in one element of an equivalent class can be generalized to all elements of that class. Ex. 
  \[ D = \{1, 2, 3, 4, 5, 6, 7\} \Rightarrow D = \{[2, 4, 6], [3, 5, 7]\}. \]
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5-queens Symmetry Example $S = 180$ Rotation

\begin{tabular}{|c|c|c|c|c|}
\hline
$x_1$ & 1 & 2 & 3 & 4 & 5 \\
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$x_2$ & 1 & 2 & 3 & 4 & 5 \\
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$x_3$ & 1 & 2 & 3 & 4 & 5 \\
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$x_4$ & 1 & 2 & 3 & 4 & 5 \\
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$x_5$ & 1 & 2 & 3 & 4 & 5 \\
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\end{tabular}
5-queens Symmetry Example $S = 180$ Rotation

- Rotate by 180 degrees.
5-queens Symmetry Example \( S = 180 \) Rotation

- Rotate by 180 degrees.
5-queens Symmetry Example $S = 180$ Rotation

- Rotate by 180 degrees.
- $x_1$ exchanges with $x_5$ and $x_2$ with $x_4$.
- New domains $\theta(val) = 6 - val$ for each $x_i$.
- Equivalence classes:
  - Variables $\{x_1, x_2\}$, $\{x_2, x_4\}$ and $\{x_3\}$.
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  - Values $\{1, 5\}$, $\{2, 4\}$, $\{3\}$.
- Reflection about the horizontal axis and vertical axis.
- Rotation by 360? Rotation by 90?
5-queens - Different Formulation

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 & 25 \\
\end{array}
\]

- \( X = \{x_1, x_2, x_3, x_4, x_5\} \)
- \( D = \{1, 2, \ldots, 25\} \)
- What are the symmetries here? Do they include domains, variables or both?
5-queens - Different Formulation

- $X = \{x_1, x_2, x_3, x_4, x_5\}$
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- $X = \{x_1, x_2, x_3, x_4, x_5\}$
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- What are the symmetries here? Do they include domains, variables or both?
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What are the symmetries here? Do they include domains, variables or both?

All 8 symmetries.
Formulation of CSP has Symmetry and not the Problem

- The definition of the symmetry applies to the definition of CSP and not to the problem itself.
- Different CSP’s formulations of the same problem can have different symmetries.
- What symmetry to select?
Formulation of CSP has Symmetry and not the Problem

- The definition of the symmetry applies to the definition of CSP and not to the problem itself.
- Different CSP’s formulations of the same problem can have different symmetries.
- What symmetry to select? What about one that produces the smallest number of equivalent classes?
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Historical Note
Three Approaches for Symmetrical CSPs

Adding symmetry breaking global constraints

- Adding global constraints to convert it to an asymmetrical CSP.

Modify search

- Pruning symmetric states as they appear in search.

Modify search heuristics

- Using symmetry-breaking rules to guide search.
Removing Symmetry from the Problem - Global Symmetry

- Puget [93] while developing PECOS tool.
- Symmetry can cause a combinatorial explosion of the search space.
- Arc-consistency AC is not adapted to symmetrical CSPs. Ex. Pigeon Hole problem.
- In symmetrical CSP a *permutation of the variables* map one solution onto another solution.
- Removing symmetrical solutions by adding a constraint - if $C \subseteq C'$ then $\text{Sol}(P') \subseteq \text{Sol}(P)$ - reduction.
- Add static symmetry breaking constraints - *an ordering constraint* $x_1 < x_2 < \cdots < x_n$ - and do AC after that.
Creating a Global Constraint

Example

- \( V = \{v_0, v_1, v_2\}, D = \{0, 1, 2\} \)
- \( C : v_0 \neq v_1 \land v_1 \neq v_2 \land v_2 \neq v_0 \)
- How many solutions?
Creating a Global Constraint

Example

- $V = \{v_0, v_1, v_2\}, D = \{0, 1, 2\}$
- $C : v_0 \neq v_1 \land v_1 \neq v_2 \land v_2 \neq v_0$
- How many solutions?
- Has a symmetry (permutation): $v_0 \rightarrow v_1, v_1 \rightarrow v_2, v_2 \rightarrow v_0$
Creating a Global Constraint

Example

- $V = \{v_0, v_1, v_2\}$, $D = \{0, 1, 2\}$
- $C : v_0 \neq v_1 \land v_1 \neq v_2 \land v_2 \neq v_0$
- How many solutions?
- Has a symmetry (permutation): $v_0 \rightarrow v_1$, $v_1 \rightarrow v_2$, $v_2 \rightarrow v_0$
- Adding $v_0 < v_1 < v_2$ - How many solutions?
General Direction

- Enforcing GAC on this global constraint reduces the problem.
- Depending on the decomposition of a problem GAC propagation can be NPC.
- In "other" constraint paper by Law at al. [CP07].
  - Proposed SigLex global constraint.
  - Its GAC propagation is \( P \).
  - But it prunes only some symmetric values in general cases.
Symmetry is Dynamic [Meseguer & Torras 2001]

- Symmetries holding at the initial states is a global symmetry.
Symmetry is Dynamic [Meseguer & Torras 2001]

<table>
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<tr>
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<td>$x_5$</td>
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- After an assignment to $v_i$ the global symmetry may break.
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- Symmetries holding at the initial states is a global symmetry.
- After an assignment to $v_i$ the global symmetry may break.
- Yet, new symmetries can appear in some states.
Symmetry is Dynamic

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- After an $v_i$ assignment the global symmetry can break.
- Yet, new symmetries can appear in some states.
- Symmetries can be broken and restored during the search.
Symmetry is Dynamic

- Symmetries holding at the initial states is a global symmetry.
- After an \( v_i \) assignment the global symmetry can break.
- Yet, new symmetries can appear in some states.
- Symmetries can be broken and restored during the search.
Pruning Symmetric States from Search

Symmetric Variables [Brown et al. 1989]

- Does not select $vvp$ if $vvp$ leads to a redundant partial assignment.
- Determines if a current partial assignment $X$ is equivalent to a smaller assignment under a symmetry group $G$.
- Has pseudo code of the Backtracking Algorithm with Symmetries.
- Symmetries are given.
Pruning Symmetric States from Search

Symmetric Values [Freuder 1991]

- Only selects one *val* from equivalence class of values during *vvp* selection.
- Values $a$ and $b$ are neighborhood interchangeable if each *vvp* is consistent with their neighborhood.
- Algorithm to determine local value interchangeability is $O(n^2d^2)$.
- Symmetries are discovered.

```
Domain

Eq. class 1  Eq. class 2  Eq. class 3
```

![Diagram showing equivalence classes and their connections]
Symmetric Variables and Values [Backofen & Will CP99, Gent & Smith 2000]

- Does not interfere with the heuristic searches (variable ordering).
- Adds symmetry breaking constraints to the right branches of search tree.

$x_1 = 2, x_2 = 3$ - backtracking
Symmetric Variables and Values [Backofen & Will CP99, Gent & Smith 2000]

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\[ x_1 = 2, x_2 = 3 - \text{backtracking} \]
\[ x_1 = 2, x_2 \neq 3 - \text{should we consider } x_4 = 3? \]
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$x_1 = 2, x_2 = 3$ - backtracking

$x_1 = 2, x_2 \neq 3$ - should we consider $x_4 = 3$? Depends if $x_5 = 5$ or not

If $x_5 \neq 5$ then $x_2 = 3$ and $x_3 = 3$ are not equivalent. Generally it is not known if $x_5 = 5$ or $x_5 \neq 5$.

Adding a conditional constraint $x_1 = 1 \land x_2 \neq 3 \land x_5 = 5 \Rightarrow x_4 \neq 3$. 
Use Symmetry to Guide Search

Dynamic Variable Ordering [Meseguer & Torras 2001]

- Direct search toward subspaces with many non-symmetric states.
- Selecting \( vvp \) that breaks the most of the symmetries.
- It will lead to more evenly distributed solutions in the CSP’s state space.
- More about it in my project presentation.
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Avoiding symmetric path in search [Glaischer 1874, Brown et al. 1989]

Value interchangeability [Freuder 1991]

Symmetry breaking constraints [Puget 93, Backofen & Will 99]

Discovering symmetries
  - Equivalent to graph isomorphism.
  - Complexity unknown (P? NPC?)
  - Discover symmetry generators with Nauty, Saucy, AUTOM

Rolf Backofen and Sebastian Will. Excluding symmetries in constraint-based search.

Darga et al. Saucy.
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