Title: A Filtering Algorithm for Constraints of Difference in CSPs
Author: J.-Ch. Régin
Proc.: AAAI 1994
Pages: 362–367

Foundations of Constraint Processing
CSCE421/821, Spring 2008
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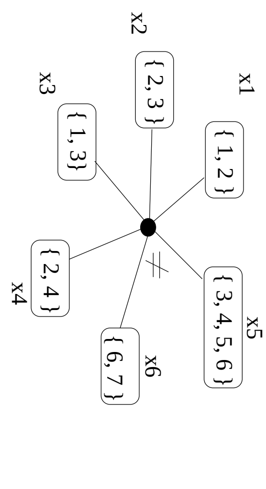
Images scanned from paper by Nimit Mehta

All-diffs constraint

Constraint: C
Variables: $X_C = \{x_1, x_2, \ldots, x_6\}$



Context: finite CSPs

Goal: efficiency of arc consistency

Focus: All-diff constraints

Result: efficient algorithm $\begin{cases} \text{Space} : \mathcal{O}(pd) \\ \text{Time} : \mathcal{O}(p^2d^2) \end{cases}$

p: #vars, d: max domain size

Application: used in RESYN for subgraph isomorphism
(plan synthesis in organic chemistry)

Contributions

- An algorithm to establish arc consistency in an all-diff constraint
  → efficient
  → powerful pruning

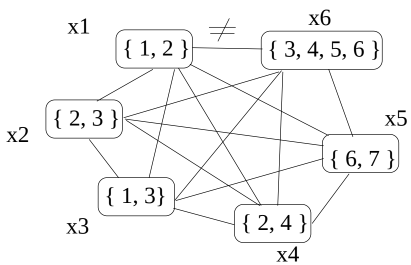
- An algorithm to propagate deletions among several all-diff constraints

- Illustration on the zebra problem

Why?

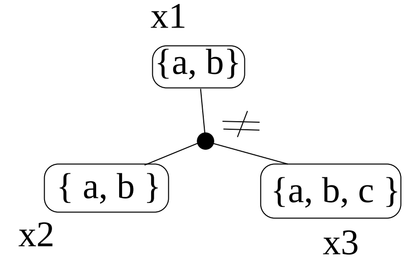
- GAC4 handles n-ary constraints
  $\rightarrow$ good pruning power
  $\rightarrow$ quite expensive:
  depends on size and number
  of all admissible tuples $= \frac{d!}{(d-p)!}$
  p: #vars, d: max domain size

- Replace n-ary by a set of binary constraints, then use AC-3 or AC-4
  $\rightarrow$ cheap
  $\rightarrow$ bad pruning



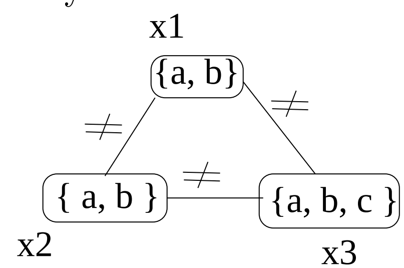
Example

- n-ary constraint



GAC4: rules out a, b for x_3

- Set of binary constraints



AC-3/4 ends with no filtering

Notations

CSP: $\mathcal{P} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$

$C \in \mathcal{C}$ defined on $X_C = \{x_{i_1}, x_{i_2}, \ldots, x_{i_j}\} \subseteq \mathcal{X}$

p: arity of C, $p = |X_C|$

d: max $|D_{x_i}|$

- A value $\underline{a_i \text{ for } x_i \text{ is consistent for } C}$, if $\exists$ values for other all variables in X_C such that these values and a_i simultaneously satisfy C
- A constraint $\underline{C \text{ is consistent}}$, if all values for all variables X_C are consistent for C
- A <u>CSP is arc-consistent</u>, if all constraints (whatever their arity) are consistent
- A <u>CSP is diff-arc-consistent</u> iff all its all-diffs constraints are arc-consistent

Value Graph

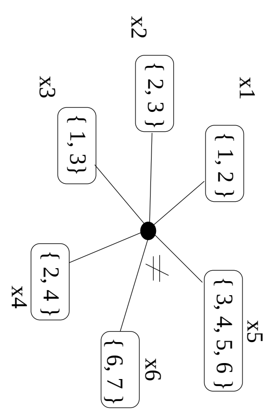
Given C, an all-diff constraint, the <u>value Graph</u> of C is a bipartite graph

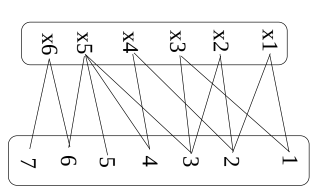
$$GV(C) = (X_C, D(X_C), E)$$

Vertices: $X_C = \{x_{i_1}, x_{i_2}, \ldots, x_{i_j}\}$

Vertices: $D(X_C) = \cup_{x \in X_C}(D_x)$

Edges: (x_i, a) iff $a \in D_x$

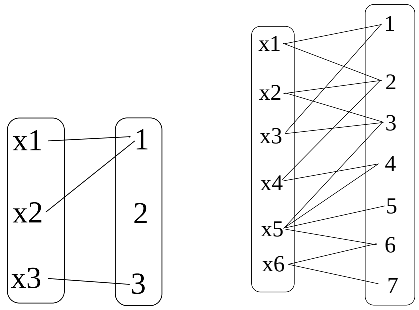




Space complexity?

Draw GV of the 3-node coloring example

Definitions: matching



Matching: a subset of edges in G with no vertex in common

Max. matching biggest possible

Matching covers a set X: every vertex in X is an endpoint for an edge in matching

- Left: M that covers X_C is a max matching
- If every edge in GV(C) is in a matching that covers X_C, C is consistent

Theorem 1

CSP: $\mathcal{P} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ is diff-arc-consistent iff

for every all-diff $C \in \mathcal{C}$

every edge GV(C) belongs to a matching
that covers X_C in GV(C)

Task:

Repeat for each all-diff constraint,
- Build G ($\equiv$ GV) of all-diff constraint C
- Remove edges that do not belong to <u>any</u> matching covering X_C

Algorithm 1:

- Compute one M(G), maximal matching in G
- If M(G) does not cover X_C, then stop
- Using M(G), remove edges that do not belong...

Algorithm 1: DIFF-INITIALIZATION(C)
% returns false if there is no solution, otherwise true
% the function COMPUTEMAXIMUMMATCHING(G) computes a maximum matching in the graph G
begin
  Build $G = (X_C, D(X_C), E)$
  $M(G) \leftarrow$ COMPUTEMAXIMUMMATCHING(G)
  if $|M(G)| < |X_C|$ **then return** *false*
  REMOVEEDGESFROMG(G,$M(G)$)
  return *true*
end

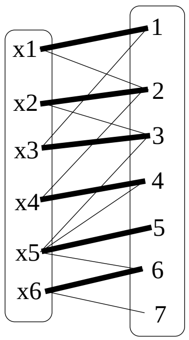
$\longrightarrow$ Hopcroft & Karp: Efficient procedure
for computing <u>**a**</u> matching covering X_C
$\longrightarrow$ Or, maximal flow in bipartite graph (less efficient)

Our problem becomes

Given:
- an all-diff constraint C
- its value graph $G = (X, Y, E)$
- one maximum covering $M(G)$

Remove edges that belong to <u>no</u> matching covering X

Definitions



Given a matching M:

matching edge: an edge in M

free edge: an edge not in M

matched vertex: incident to a matching edge

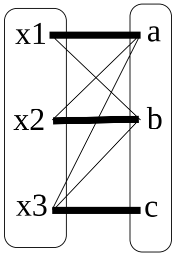
free vertex: otherwise

alternating path (cycle): a path (cycle) whose edges are alternatively matching and free

length of a path: number of edges in path

vital edge: belongs to every maximum matching

Questions



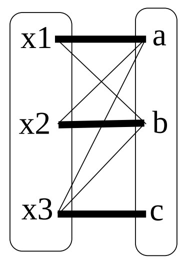
Indicate:

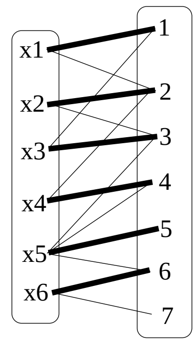
- matching edges
- free edges
- matched vertices
- a free vertex
- an alternating path, length?
- an alternating cycle, length?
- a vital edge

Property 1 (Berge)

An edge belongs to some of but not all maximum matchings, iff for an arbitrary maximum matching M, it belongs to either:

- an even alternating cycle, or
- an even alternating path that begins at a free vertex

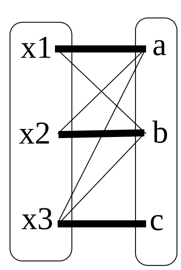


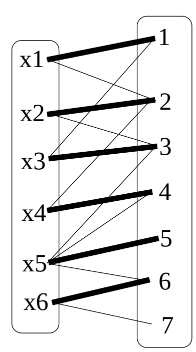


Thus:

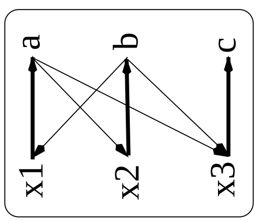
The edges to remove should not be in:

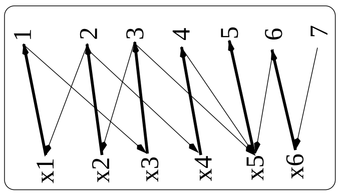
- all matchings (vital)
- an even alternating path starting at a free vertex
- an even alternating cycle

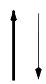




Given:
- $G = (X, Y, E)$
- a matching $M(G)$ covering X
- Build G_O, by orienting the edges







- every directed cycle in G_O corresponds to an even alternating cycle of G, and conversely
- every directed simple path in G_O, starting at a free vertex corresponds to an even alternating path of G starting at a free vertex, an conversely

Task:

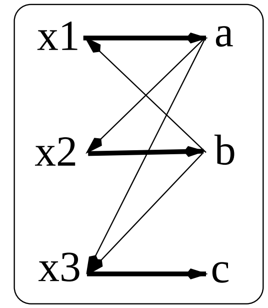
Given G, and $M(G)$, remove edges that do not belong to any matching covering X_C

Algorithm 2

- Build G_O
- Mark all edges of G_O as unused
- Identify all directed edges that belong to a directed simple path starting at a free vertex by a breadth-first search, mark them as used
- Compute strongly connected components in G_O. Mark "used" any directed edge between two vertices in the same strongly connected component, as any such edge belongs to a directed cycle and conversely
- All remaining unused edges,
  if they are in $M(G)$, mark them as vital
  else put them in RE and remove them from G

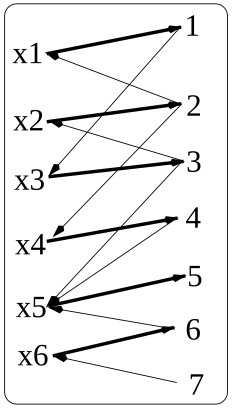
Algorithm 2

Algorithm 2: REMOVEEDGESFROMG(G,$M(G)$)
% RE is the set of edges removed from G.
% $M(G)$ is a matching of G which covers X
% The function returns RE
begin
1 Mark all directed edges in G_O as "unused".
Set RE to $\emptyset$.
2 Look for all directed edges that belong to a directed simple path which begins at a free vertex by a breadth-first search starting from free vertices, and mark them as "used".
3 Compute the strongly connected components of G_O. Mark as "used" any directed edge that joins two vertices in the same strongly connected component.
4 **for** each directed edge de marked as "unused" **do**
  set e to the corresponding edge of de
  if $e \in M(G)$ **then** mark e as "vital"
  else
    $RE \leftarrow RE \cup \{e\}$
    remove e from G
return RE
end



Algorithm 2

- ...
- Identify all edges starting at a free vertex by a breadth-first search, mark them as used
- Compute strongly connected components in G_O. Mark "used" any directed edge between two vertices in the same strongly connected component, as any such edge belongs to a directed cycle and conversely
- All remaining unused edges,
  if they are in $M(G)$, mark them as vital
  else put them in RE and remove them from G



Algorithm 2

- ...
- Identify all edges starting at a free vertex by a breadth-first search, mark them as used
- Compute strongly connected components in G_O. Mark "used" any directed edge between two vertices in the same strongly connected component, as any such edge belongs to a directed cycle and conversely
- All remaining unused edges,
  if they are in $M(G)$, mark them as vital
  else put them in RE and remove them from G

So far..

Given C, remove edges that are not consistent for C

.. but,

A variable x may be in more than one all-diff constraints,
i.e. x may be in X_{C_i} and X_{C_j}, with C_i and C_j two all-diff constraints

How to propagate the effect of filtering of C_i on C_j?

→ start from scratch?

→ propagate deletions more intelligently
use the fact that before deletion due to C_j,
a matching covering X_{C_i} was known in GV(C_i)

Assume we have C_i, C_j, and C_k involving a given variable

$$\text{Compute} \begin{cases} \text{RE}(C_i), \text{RE}(C_j), \text{RE}(C_k), \\ \text{G=GV}(C_i), \text{M(G), etc.} \end{cases}$$

Idea

Consider C_i

First remove from G deletions due to C_j, C_k

Second, try to extend the remaining edges in M(G) into a matching that covers X_{C_i}

Finally, apply **Algorithm 2**

... iterate

Consider C_i, G = GV(C_i), M(G)

Set RE $\leftarrow$ RE(C_i)

ER $\leftarrow$ RE(C_j) $\cup$ RE(C_k)

Algorithm 3: DIFF-PROPAGATION(G,$M(G)$,ER,RE)
% the function returns false if there is no solution
% G is a value graph
% $M(G)$ is a matching which covers X_C
% ER is the set of edges to remove from G
% RE is the set of edges that will be deleted by the filtering

```
begin
    computeMatching ← false
    for each e ∈ ER do
        if e ∈ M(G) then
            M(G) ← M(G) − {e}
            if e is marked as "vital" then return false
            else computeMatching ← true
        remove e from G
    if computeMatching then
        if ¬ MATCHINGCOVERINGX(G,M(G),M') then
            return false
        else
            M(G) ← M'
    RE ← REMOVEEDGESFROMG(G,M(G))
    return true
end
```

Example: the Zebra problem

5 houses of different colors
5 inhabitants, different nationalities, different pets, different drinks, different cigarettes

Consider the following facts:

1. The Englishman lives in the red house
2. The Spaniard has a dog
3. Coffee is drunk in the green house
4. The Ukrainian drinks tea
5. The green house is immediately to the right of the ivory house
6. The snail owner smokes Old-Gold
7. *etc.*

Query: who drinks water?
who owns a zebra?

Zebra: formulation

25 variables:

- 5 house-color $C_1, C_2, \ldots, C_5$
- 5 nationalities $N_1, N_2, \ldots, N_5$
- 5 drinks $B_1, B_2, \ldots, B_5$
- 5 cigarettes $T_1, T_2, \ldots, T_5$
- 5 pets $A_1, A_2, \ldots, A_5$

C_1 red	B_1 coffee	N_1 Englishman	T_1 Old-Gold	A_1 dog
C_2 green	B_2 tea	N_2 Spaniard	T_2 Chesterfield	A_2 snails
C_3 ivoiry	B_3 milk	N_3 Ukranian	T_3 Kools	A_3 fox
C_4 yellow	B_4 orange	N_4 Norwegian	T_4 Lucky-Strike	A_4 horse
C_5 blue	B_5 water	N_5 Japanese	T_5 Parliament	A_5 zebra

Domain of each variable $= \{1, 2, 3, 4, 5\}$
$(\equiv \{h1, h2, h3, h4, h5\})$
Constraints 2–15?

Formulating Constraint 1:

1. Binary constraint between any pair in each cluster: binary CSP
2. Five 5-ary all-diff constraints: non-binary CSP
3. The 5-ary constraints are replaced with their GV. Space?

Results (I)

Formulation 1 solved with AC

2 (=)		3 (=)		4 (=)		5 (=)		6 (-1)		7 (=)		8 (=)		9
N_1	C_1	N_2	A_1	B_1	C_2	N_3	B_2	C_2	C_3	T_1	A_2	T_3	C_4	B_3
3	3	2	2	4	4	2	2	4	3	1	1	1	1	3
4	4	3	3	5	5	4	4	5	4	2	2	3	3	
5	5	4	4			5	5			3	3	4	4	
		5	5							4	4	5	5	
										5	5			

10	11 (±1)		12 (±1)		13 (=)		14 (=)		15 (±1)			
N_4	T_2	A_3	A_4	T_3	T_4	B_4	N_5	T_5	N_4	C_5	A_5	B_5
1	1	2	2	1	1	1	2	2	1	2	1	1
	2	1	2	3	2	2	3	3			2	2
	2	3	3	4	4	4	4	4			3	4
	3	2	4	3	5	5	5	5			4	5
	3	4	4	5							5	
	4	3	5	4								
	4	5										
	5	4										

Figure 4.

Formulation 2 solved with GAC-4

2 (=)		3 (=)		4 (=)		5 (=)		6 (-1)		7 (=)		8 (=)		9
N_1	C_1	N_2	A_1	B_1	C_2	N_3	B_2	C_2	C_3	T_1	A_2	T_3	C_4	B_3
3	3	3	3	4	4	2	2	4	3	3	3	1	1	3
4	4	4	4	5	5	4	4	5	4	4	4			
5	5	5	5			5	5			5	5			

10	11 (±1)		12 (±1)		13 (=)		14 (=)		15 (±1)			
N_4	T_2	A_3	A_4	T_3	T_4	B_4	N_5	T_5	N_4	C_5	A_5	B_5
1	2	1	2	1	2	2	2	2	1	2	1	1
	2	3			4	4	3	3			3	
	3	4			5	5	4	4			4	
	4	3					5	5			5	
	4	5										
	5	4										

Figure 5.

Formulation 3 solved with the new technique.

Same results as 2.

Results (II)

a: # of binary constraints

p: size of a cluster

c: # of clusters

d: # of values in a domain

$\mathcal{O}(ad^2)$: complexity of AC on binary

Formulation 1 solved with AC

- number of binary constraint added is $\mathcal{O}(cp^2)$
- filtering complexity is $\mathcal{O}((a + cp^2)d^2)$

Formulation 2 solved with GAC-4

- filtering complexity is $\mathcal{O}(\frac{d!}{(d-p)!}p)$

Formulation 3 solved with the new technique

- arc-consistency is $\mathcal{O}(ad^2)$
- all-diff filtering is $\mathcal{O}(cp^2d^2)$
- total filtering is $\mathcal{O}(ad^2 + cp^2d^2)$

Extension

Improved bounds by J.-F. Puget (AAAI 99) for ordered domains (*e.g.*, time in scheduling).

Lesson

We can improve the performance of search by:

- identifying special structures in the constraint graph (*e.g.*, tree, biconnected components, DAG)
- identifying special types of constraints (*e.g.*, functional, anti-functional, monotonic, all-diffs)

Improved arc-consistency

Van Hentenryck et al. AIJ 92

Functional

A constraint C is functional with respect to a domain D iff for all $v \in D$ (respectively $w \in D$) there exists at most one $w \in D$ (respectively $v \in D$) such that $C(v, w)$.

Anti-functional

A constraint C is anti-functional with respect to a domain D iff $\neg C$ is functional with respect to D.

Monotonic

A constraint C is monotonic with respect to a domain D iff there exists a total ordering on D such that, for all values v and $w \in D$, $C(v, w)$ holds implies $C(v', w')$ holds for all values v' and $w' \in D$ such that $v' \leq v$ and $w' \leq w$.