CSE235: Recitation of Week 6

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Problem: A proof by contradiction, using cases

Prove that if n is a positive integer, then $n^2 + 3n + 2$ is even.

Answer: The proof is done using contradiction and cases.

The statement to prove is the following:

> If n is a positive integer, then $n^2 + 3n + 2$ is even.

It is of the form $p \to q$, whose negation is $p \wedge \neg q$. Thus, we assume that:

- n is positive integer (which p), and
- $n^2 + 3n + 2$ is odd (which $\neg q$).

Then, we show that the above yields a contradiction, which will prove the original statement.

$n^2 + 3n + 2$ is odd $\Rightarrow$

$n^2 + 3n + 2$ can be written as $2m + 1$ for some integer m ($m \in \mathbb{Z}$) $\Rightarrow$

$n^2 + 3n + 2 = 2m + 1 \Rightarrow$

$n^2 + 3n = 2m - 1$ (moving 2 to the righ-hand side) $\Rightarrow$

$n^2 + 3n - 2m = -1$ (moving 2m to the left-hand side) $\Rightarrow$

$n(n + 3) - 2m = -1$ (factoring out n)

Case 1: n is odd. n is odd $\Rightarrow$ $(n + 3)$ is even $\Rightarrow$ $n(n + 3)$ is even.

Case 2: n is even. n is even $\Rightarrow$ $n(n + 3)$ is even

Thus, in all possible cases, $n(n + 3)$ is even $\Rightarrow$ $n(n + 3)$=$2k$ for some $k \in \mathbb{Z}$.

From above, we have $n^2 + 3n + 2$ is odd $\Rightarrow$

$n(n+3) - 2m = -1 \Rightarrow$

$2k - 2m = -1 \Rightarrow$

$2(k-m) = -1$ where $(k-m) \in \mathbb{Z}$, which is a contradiction be it says that an even integer is equal -1.

We have shown that the negation of the statement to prove yields a contradiction. In conclusion, the statement holds. $\square$

Problem: Proof by cases. Prove that $n^4 - n^2$ is divisible by 3 for all $n \in N$.

Answer: To show that $n^4 - n^2$ is divisible by 3, we consider three cases:

1. $n = 3k$,
2. $n = 3k + 1$, and
3. $n = 3k + 2$,

where $k \in N$. The three cases cover all possible numbers in N. So, we will show the proof by cases.

Case 1: $n = 3k$**.**

$$\begin{aligned} n^4 - n^2 &= n^2(n^2 - 1) \\ &= (3k)^2((3k)^2 - 1) \\ &= 3(3k^2((3k)^2 - 1)). \end{aligned}$$

Since this number is a multiple of 3, it is divisible by 3. Thus, we have shown that for $n = 3k$, $n^4 - n^2$ is divisible by 3.

Case 2: $n = 3k + 1$**.**

$$\begin{aligned} n^4 - n^2 &= n^2(n^2 - 1) \\ &= (3k+1)^2((3k+1)^2 - 1) \\ &= (3k+1)(3k+1)((3k+1)(3k+1) - 1) \\ &= (9k^2 + 6k + 1)(9k^2 + 6k + 1 - 1) \\ &= (9k^2 + 6k + 1)(9k^2 + 6k) \\ &= (9k^2 + 6k)(9k^2 + 6k + 1) \\ &= 3(3k^2 + 2k)(9k^2 + 6k + 1). \end{aligned}$$

Since this number is a multiple of 3, it is divisible by 3. Thus, we have shown that for $n = 3k + 1$, $n^4 - n^2$ is divisible by 3.

Case 3: $n = 3k + 2$.

$$\begin{aligned} n^4 - n^2 &= n^2(n^2 - 1) \\ &= (3k+2)^2((3k+2)^2 - 1) \\ &= (3k+2)(3k+2)((3k+2)(3k+2) - 1) \\ &= (9k^2 + 12k + 4)(9k^2 + 12k + 4 - 1) \\ &= (9k^2 + 12k + 4)(9k^2 + 12k + 3) \\ &= (9k^2 + 12k + 3)(9k^2 + 12k + 4) \\ &= 3(3k^2 + 4k + 1)(9k^2 + 12k + 4). \end{aligned}$$

Since this last number is a multiple of 3, it is divisible by 3. Thus, we have shown that for $n = 3k + 2$, $n^4 - n^2$ is divisible by 3.

Therefore, we have shown that for all cases, $n^4 - n^2$ is divisible by 3.

Problem: Problem 2.1.9 from textbook

Determine whether each of these statement is T or F.

a) $x \in \{x\}$

b) $\{x\} \subseteq \{x\}$

c) $\{x\} \in \{x\}$

Answer: Answer is in textbook.

Problem: Problem 2.1.31 from textbook

Explain why A × B × C and (A × B) × C are not the same.

Answer: Answer is in textbook.

Problem: Problem 2.2.26 from textbook

Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

1. $A \cap (B \cup C)$
2. $\bar{A} \cap \bar{B} \cap \bar{C}$
3. $(A - B) \cup (A - C) \cup (B - C))$

Answer: Answer is in the handwritten recitation notes of Week7 of the GTA.

Problem: Problem 2.2.32 from textbook

Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

Answer: Answer is in the handwritten recitation notes of Week7 of the GTA.

Problem: Problem 2.2.40 from textbook

Determine whether the symmetric different is associative; that is, if $A, B, andC$ are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

Answer: Answer is in the handwritten recitation notes of Week7 of the GTA.