Recitation # 8. Oct. 14. 2008.

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1. Logic. (Orsoh, HW1. Prob. 7).

* Find propositions logically equivalent to the following using only the connectives $\neg$ and $\vee$.

a) $p \leftrightarrow q$

solution: $p \leftrightarrow q$

$\Rightarrow [(p \to q) \wedge (q \to p)]$ Definition of Equivalence

$\Rightarrow [(\neg p \vee q) \wedge (\neg q \vee p)]$ Implication.

$\Rightarrow \neg\neg[(\neg p \vee q) \wedge (\neg q \vee p)]$ Double Negation.

$\Rightarrow \neg[\neg(\neg p \vee q) \vee \neg(\neg q \vee p)]$ De Morgan.

c) $(p \to q) \wedge (q \vee r)$

solution $(p \to q) \wedge (q \vee r)$

$\Rightarrow [(\neg p \vee q) \wedge (q \vee r)]$ Implication

$\Rightarrow \neg\neg[(\neg p \vee q) \wedge (q \vee r)]$ Double Negation.

$\Rightarrow \neg[\neg(\neg p \vee q) \vee \neg(q \vee r)]$ De Morgan.

1.4.46. a). Determine the truth value of the statement $\exists x \forall y \, (x \leq y^2)$. if the domain for the variables consists of

a) the positive real numbers.

Solution: F. $\because$ no matter how small a positive number x we choose, if we let $y = \sqrt{\frac{x}{2}}$, then, $x = 2y^2$, and it will not be true that $x \leq y^2$.

Exe:

P107 28: Express this statement using quantifiers:

"There is a building on the campus of some college in the United States in which every room is painted white."

Solution: $W(r)$: room r is painted white.

$I(r,b)$: room r is in building b.

$L(b,u)$: ~~mean that~~ building b is on the campus of college u.

The statement is: there is some u, & some building on the campus of u such that every room in b is painted white.

$\Downarrow$

$$\exists u \, \exists b \left(L(b,u) \wedge \forall r \left(I(r,b) \rightarrow W(r) \right) \right)$$

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* Find a counter example to prove that the following logical implication is false.

$$\exists x\,[p(x) \vee q(x)] \Rightarrow \exists x\,[p(x) \wedge q(x)]$$

- Suppose that a and b are the only elements in the universe of discourse x.

Suppose $\begin{cases} P(a) \text{ is True} \\ q(a),\ q(b),\ \text{and } p(b) \text{ are False.} \end{cases}$

Now, we can see that the left hand side is True, since there is one element in x makes P(x) True.

However, the right hand side is false. We can not find an element in x that satisfies both P(x) and q(x)!

So, the logical implication is false.

2. Formal Proof.

* Use propositional logic and predicate logic to prove that : if $G(a)$, $\forall x\,(G(x) \to S(x))$, $\forall y\,(P(y) \to \neg G(y))$,

then $\exists z\,(S(z) \wedge \neg P(z))$.

for $x, y, z \in$ same universe of discourse.

proof :

1.	$G(a)$	Hypothesis
2.	$\forall x\,(G(x) \to S(x))$	Hypothesis
3.	$\forall y\,(P(y) \to \neg G(y))$	Hypothesis
4.	$G(a) \to S(a)$	Universal Instadiation from 2.
5.	$S(a)$	modus Ponens from 1. 4.
6.	$P(a) \to \neg G(a)$	Universal Instantiation from 3
7.	$\neg P(a)$	modus tollens from 1. 6.
8.	$S(a) \wedge \neg P(a)$	Conjunction from 5. 7.
9.	$\exists z\,(S(z) \wedge \neg P(z))$.	

* Give a formal proof for each of the following statements.

a) If $r \to q$, $\neg(q \vee p)$

then $\neg(r \vee p)$

Proof: (by contradiction)

1.	$r \to q$	Hypothesis
2.	$\neg(q \vee p)$	Hypothesis
3.	$\neg\neg(r \vee q)$	negation of conclusion.
4.	$r \vee q$	Double negation from 3.
5.	$\neg q \wedge \neg p$	De Morgan from 2.
6.	$\neg q$	simplification from 5
7.	$\neg p$	" " "
8.	r	disjunctive syllogism from 4. 7
9.	q	modus ponens from 1. 8.
10	$\neg q \wedge q$	conjunction 9. 6.
11.	Contradiction.	

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* show that there are no integer solutions to the following:

$x^2 + 4y^2 = 23$.

proof: we proof it by ~~[illegible]~~ contradiction.

We assume that there is an integer solution for the equation above, i.e. $a^2 + 4b^2 = 23$.

case 1: if $\begin{cases} a = 2m \\ b = 2n \end{cases}$ then. $4m^2 + 4(4n^2) = 23$.

$2[2m^2 + 2(4n^2)] = 23$

Since left hand side is even. and right hand side is odd.

$\therefore$ It is a contradiction!

Case 2: if $\begin{cases} a = 2m+1 \\ b = 2n \end{cases}$ then $(2m+1)^2 + 4(2n)^2 = 23$

$4m^2 + 4m + 1 + 4(4n^2) = 23$

$4m^2 + 4m + 4 \times 4n^2 = 22$

$2m^2 + 2m + 2 \times 4n^2 = 11$

$\because$ left hand side is even, and right hand side is odd.

$\therefore$ it is a contradiction!

case 3: if $\begin{cases} a = 2m \\ b = 2n+1 \end{cases}$ then

$4m^2 + 4(4n^2 + 4n + 1) = 23$

$2[2m^2 + 2(4n^2 + 4n + 1)] = 23$

$\because$ left hand side is even, and right hand side is odd.

$\therefore$ it is a contradiction!

case 4: if $\begin{cases} a=2m+1 \\ b=2n+1 \end{cases}$ then,

$$4m^2+4m+1+4(4n^2+4n+1)=23$$
$$4m^2+4m+4(4n^2+4n+1)=22$$
$$2m^2+2m+2(4n^2+4n+1)=11$$
$$2[m^2+m+(4n^2+4n+1)]=11$$

$\because$ Left hand side is even, and right hand side is odd.

$\therefore$ it is contradiction!

From Case 1 to Case 4. we can proof that $x^2+4y^2=23$ has no integer solution.

3. Set.

* For the following statement, Determine whether it is true or false.

$$A \cap B = (A \cup B) - [(A \cap \bar{B}) \cup (\bar{A} \cap B)]$$

proof: first we choose any $x \in [(A \cup B) - [(A \cap \bar{B}) \cup (\bar{A} \cap B)]]$

$\Rightarrow x \in (A \cup B) \wedge x \notin (A \cap \bar{B}) \cup (\bar{A} \cap B)$

$\Rightarrow x \in (A \cup B) \wedge x \notin (A \oplus B)$

$\Rightarrow x \in (A \cap B)$

So, $(A \cup B) - [(A \cap \bar{B}) \cup (\bar{A} \cap B)] \leq A \cap B$

Second, we choose any $x \in A \cap B$

$\Rightarrow x \in [A \cap B]$

$\Rightarrow x \in [(A \cup B) - (A \oplus B)]$

$\Rightarrow x \in [(A \cup B) - [(A - B) \cup (B - A)]]$

$\Rightarrow x \in (A \cup B) - [(A \cap \bar{B}) \cup (\bar{A} \cap B)]$

So, ~~A~~ $A \cap B \leq (A \cup B) - [(A \cap \bar{B}) \cup (\bar{A} \cap B)]$

So. $A \cap B = (A \cup B) - [(A \cap \bar{B}) \cup (\bar{A} \cap B)]$

4. Function:

* Given functions mapping $\mathbb{R}$ to $\mathbb{R}$, as follows:

$f(x) = 3 - x$

$g(x) = \frac{2}{x}$ Find. $f \circ g \circ h$.

$h(x) = 3x^2 - x$

solution: $f \circ g \circ h(x)$

$= f(g(h(x))) = f(g(3x^2 - x))$

$= f\left(\frac{2}{3x^2 - x}\right) = 3 - \frac{2}{3x^2 - x}$

* Let $A = \{x \mid x \neq \frac{1}{2}\}$ and define $f: A \to \mathbb{R}$.
by $f(x) = \frac{4x}{2x-1}$. Proof that f is one-to-one.
Find the inverse for $f: A \to B$ ($B = \{y \in \mathbb{R} \mid y \neq 2\}$)

Solution:

① Suppose $f(a_1) = f(a_2)$,

by definition of $f(x) \Rightarrow \frac{4a_1}{2a_1 - 1} = \frac{4a_2}{2a_2 - 1}$.

so. $4a_1(2a_2 - 1) = 4a_2(2a_1 - 1)$

$\Rightarrow 8a_1a_2 - 4a_1 = 8a_1a_2 - 4a_2$

$\Rightarrow 8a_1a_2 - 8a_1a_2 = 4a_1 - 4a_2$

$\Rightarrow a_1 - a_2 = 0$

$\Rightarrow a_1 = a_2$.

Thus, f is one-to-one.

② Given that $B = \{y \in \mathbb{R} \mid y \neq 2\}$,

suppose that $f(x) = \frac{4x}{2x-1} = y$.

To find the inverse, we need to find $f^{-1}(y) = x$.

so, we need to solve for x,

As shown above, we know that

$$(2x-1)\,y = 4x$$

$$\Rightarrow \; 2xy - y - 4x = 0$$

$$\Rightarrow \; 2x\,(y - 2) = y.$$

$$\Rightarrow \; 2x = \frac{y}{y-2}$$

$$\Rightarrow \; x = \frac{y}{2(y-2)}$$

Thus, the inverse function is

$$f^{-1}(y) = \frac{y}{2(y-2)} \quad \text{where } \{y \in \mathbb{R} \mid y \neq 2\}.$$

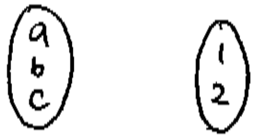
* Let $S = \{1, 2\}$ and $T = \{a, b, c\}$

a) How many ONTO functions are there mapping $S \to T$?

Solu: 0 : since the number of element in S is less than the number of element in T, it is impossible to map all element in T without having an element in S mapping to more than 1 element in T.

b) How many ONTO functions are there mapping $T \to S$?

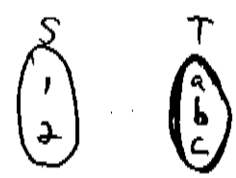
Solution: $2^3 - 2 = 6$



c) How many one-to-one functions are there mapping $S \to T$?

Solution:

$P_3^2 = 6$ [3 choose 2]



(1, a) (2, b) (2, a) (1, b) (1, b) (2, c)
(1, a) (2, c) (2, a) (1, c) (1, c) (2, b)

d) How many one-to-one functions are there mapping $T \to S$?

solution: 0