Recitation 14

- recurrence relation
- master theorem.
- Combinatorics

7.1.23

a) Find a recurrence relation for the number of bit strings of length n that contain a pair of ~~consec~~ consecutive 0s.

Solution: Let a_n be the number of bit strings of length n containing 2 consecutive 0s.

In order to construct a bit string of length n containing 2 consecutive 0s, we could start with 1, and follows with a string of length "$n-1$" containg 2 0s. or, we could start with a 01 and follow with a string of length "$n-2$" containing 2 consecutive 0s. or we could start with a 00 and follow with any string of length $n-2$.

These three cases are mutually exclusive and exhaust the possibilit~~y~~ies for how the string might start.

From this analysis we can immediatly write down the recurrence relation. valid for $n \geq 2$:

$$a_n = a_{n-1} + a_{n-2} + 2^{n-2}$$

b) What are the initial conditions?

Solution: Since, there are no bit strings with length 0 or 1 with 2 ~~consecued~~ consecutive 0s,

The initial conditions are $a_0 = a_1 = 0$

c) How many bit strings of length 7 contain 2 consecutive 0s?

Solution: We will comput a_2 through a_7 using the recurrence relation:

$$a_2 = a_0 + a_1 + 2^{2-2} = a_0 + a_1 + 1 = 1$$
$$a_3 = a_1 + a_2 + 2^{3-2} = 0 + 1 + 2^1 = 3$$
$$a_4 = a_2 + a_3 + 2^{4-2} = 1 + 3 + 2^2 = 8$$
$$a_5 = a_3 + a_4 + 2^{5-2} = 3 + 8 + 2^3 = 19$$
$$a_6 = a_5 + a_4 + 2^{6-2} = 19 + 8 + 2^4 = 43$$
$$a_7 = a_5 + a_6 + 2^{7-2} = 19 + 43 + 2^5 = \boxed{}$$

7.1. 27.

a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or 2 stairs at a time.

Solution:

- Let a_n be the number of ways to climb n stairs.

In order to climb n stairs, a person must either start with a step of one stair and then climb n-1 stairs or start with a step of 2 stairs and then climb n-2 stairs

From above analysis, the recurrence relation is

$$a_n = a_{n-1} + a_{n-2}, \text{ for } n \geq 2$$

b) What are the initial conditions?

Solution: The initial conditions are $a_0 = 1, a_1 = 1$, since there is one way to climb no stairs (do nothing). and only one way to climb one stair.

c) How many ways can this person climb a flight of eight stairs?

Solution:

$a_8 = ?$

each term in the sequence $\{a_n\}$ is the sum of the previous 2 terms,

So,

$a_2 = a_0 + a_1 =$ $\quad$ $a_6 = a_5 + a_4 =$

$a_3 = a_1 + a_2 =$ $\quad$ $a_7 = a_6 + a_5 =$

$a_4 = a_3 + a_2 =$ $\quad$ $a_8 = a_6 + a_7 =$

$a_5 = a_4 + a_3 =$

7.2.19. Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.

Solution: The characteristic equation of this recurrence relation is:

$$r^3 + 3r^2 + 3r + 1 = 0$$

$$\Rightarrow \quad (r+1)^3 = 0$$

$$\Rightarrow \quad r_1 = r_2 = r_3 = -1$$

So. $a_n = \alpha_1 (-1)^n + \alpha_2\, n(-1)^n + \alpha_3\, n^2(-1)^n$

with initial conditions:

$$a_0 = \alpha_1 = 5$$

$$a_1 = \alpha_1(-1) + \alpha_2(-1) + \alpha_3(-1) = -9$$

$$a_2 = \alpha_1 + 2\alpha_2 + 4\alpha_3 = 15$$

$$\Rightarrow \begin{cases} \alpha_1 = 5 \\ \alpha_2 = 3 \\ \alpha_3 = 1 \end{cases}$$

$$\Rightarrow \quad a_n = (n^2 + 3n + 5)(-1)^n.$$

7.2.23. Consider the non homogeneous linear recurrence relation

$$a_n = 3a_{n-1} + 2^n.$$

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

proof: the right hand side is:

$$3 \times (-2^n) + 2^n$$
$$= -3\, 2^n + 2^n$$
$$= -2 \times 2^n$$
$$= -2^{n+1}$$

and the left hand side is $a_n = -2^{n+1}$

So $a_n = -2^{n+1}$

b) Use Theorem 5 to find all solutions of this recurrence relations.

Solution: $a_n = 3a_{n-1} + 2^n$ is non homogeneous linear recurrence relation. its associated homogeneous relation is

$$a_n = 3a_{n-1}$$

And its solutions are $a_n^{(h)} = \alpha 3^n$.

Because $F(n) = 2^n$, a reasonable trial solution is (-2^{n+1})

Thus, the general solution of this is $a_n = \alpha 3^n - 2^{n+1}$

c) Find the solution with $a_0 = 1$

$$a_0 = \alpha \cdot 3^0 - 2^1 = 1$$

$$\Rightarrow \quad \alpha - 2 = 1 \Rightarrow \alpha = 3$$

So. $a_n = 3 \cdot 3^n - 2^{n+1} = 3^{n+1} - 2^{n+1}$

Master Theorem

Section 7.3

1. $T(n) = 3T(n/2) + n^2$

Solution: $\Rightarrow \begin{cases} a = 3 \\ b = 2 \\ c = 1 \\ d = 2 \end{cases}$

$\therefore \; a = 3 \qquad b^d = 2^2 = 4$

$\rightarrow \; a < b^d$

Then, $T(n) \in \Theta(n^d) = \Theta(n^2)$

2. $T(n) = 3T(n/2) + n \log n$

Solution: $\begin{cases} a = 3 \\ b = 2 \end{cases}$ but $f(n)$ is not polynomial,

However, $f(n) = n \log n$, $f(n) \in \Theta(n \log n)$, so $k = 1$

$$T(n) \in \Theta(n^{\log_2 3} \log^{k+1} n)$$
$$\in \Theta(n^{\log_2 3} \log^2 n)$$

3. $T(n) = 0.5\,T(n/2) + n$

Solution: Theorem dose not apply,

Since $a = 0.5 < 1$

7.5.5. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if

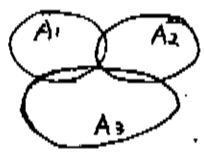
a) the sets are pairwise disjoint

solution:

Since 3 sets are pairwise disjoint, thus, $|A_1 \cup A_2 \cup A_3|$ $= |A| + |A_2| + |A_3| = 300$.

b) there are 50 common elements in each pair of sets and no elements in all three sets.

Solution:



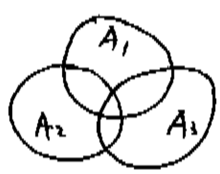
so: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1A_2| - |A_2A_3| - |A_1A_3| + |A_1A_2A_3|$

$= 100 + 100 + 100 - 50 - 50 - 50 + 0$

$= 150$

c) there are 50 common elements in each pair of sets and 25 elements in all three sets.

Solution:



so:

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1A_2| - |A_2A_3| - |A_1A_3| + |A_1A_2A_3|$

$= 100 + 100 + 100 - 50 - 50 - 50 + 25$

$= 175$

5.2.16: Pigeonhole Principle.

How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers adds up to 16 ?

Solution :

① We group the numbers into pairs that adds up to 16, that is :

$\{1, 15\}$ $\{3, 13\}$ $\{5, 11\}$ and $\{7, 9\}$.

② According to pigeonhole principle, if we choose 5 numbers from the set, then at least 2 of them must fall within the same subset, since there are only 4 subsets.

Two numbers in the same subset are the desired pair that adds up to 16.