Recitation 13:

4.1.7. Prove that $3 + 3\cdot 5 + 3\cdot 5^2 + \cdots + 3\cdot 5^n = 3\cdot(5^{n+1}-1)/4$

whenever n is a nonnegative integer.

proof: Let $P(n)$ be "$\sum_{j=0}^{n} 3\cdot 5^j = \frac{3(5^{n+1}-1)}{4}$".

Basic step: $P(0)$ is true because

$$\sum_{j=0}^{0} 3\cdot 5^j = 3 = \frac{3(5^1-1)}{4}$$

Inductive step: Assume that $\sum_{j=0}^{k} 3\cdot 5^j = \frac{3(5^{k+1}-1)}{4}$

Then, $\sum_{j=0}^{k+1} 3\cdot 5^j = \sum_{j=0}^{k} 3\cdot 5^j + 3\cdot 5^{k+1}$

$$= \frac{3\cdot(5^{k+1}-1)}{4} + 3\cdot 5^{k+1}$$

$$= \frac{3\cdot 5^{k+1} - 3 + 12\cdot 5^{k+1}}{4}$$

$$= \frac{3(5\cdot 5^{k+1}-1)}{4}$$

$$= \frac{3(5^{k+2}-1)}{4}$$

We have completed the basis step and the inductive step. Thus, by mathematical induction $P(n)$ is true ~~for~~ for all nonnegative integers. That is, the equality

$$3 + 3\cdot 5 + \cdots + 3\cdot 5^n = \frac{3(5^{n+1}-1)}{4}$$ is valid.

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Following problems based on (Rosen 2003, Gossett 2003, Hein 2002, Goodaire and Parmenter 2002, Ross and Wright 1988)

1. Prove that

$$\frac{1}{1*5} + \frac{1}{5*9} + \frac{1}{9*13} + \cdots + \frac{1}{(4n-3)*(4n+1)} = \frac{n}{4n+1} \quad \text{for all } n \geq 1.$$

Proof: First, we show that the basis is true. when $n=1$

$$\frac{1}{(4n-3)(4n+1)} = \frac{1}{1\cdot 5} = \frac{1}{5}$$

$$\frac{n}{4n+1} = \frac{1}{4*1+1} = \frac{1}{5}$$

~~when~~ we have shown that the basis step is true. by inspection.

Second, we show that the induction step is true. Assume that P(k) is true:

$$\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \cdots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

Now, we need to show that P(k+1) is true given that the above is true:

$$P(k+1) = P(k) + \frac{1}{(4(k+1)-3)(4(k+1)+1)}$$

$$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2+5k+1}{(4k+1)(4k+5)} = \frac{\cancel{(4k+1)}(k+1)}{\cancel{(4k+1)}(4k+5)}$$

$$= \frac{k+1}{4k+5}.$$

Thus, we show that P(k+1) is true. Therefore, we have shown the problem statement.

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2. Prove by induction that $\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n}$ for all $n \in \mathbb{Z}^+$, $n \geq 2$.

Proof: First, we show that the basis is true. When $n=2$,

$$\prod_{k=2}^{2}\left(1-\frac{1}{k^2}\right) = 1-\frac{1}{2^2} = 1-\frac{1}{4} = \frac{3}{4}$$

$$\frac{n+1}{2n} = \frac{2+1}{2\times 2} = \frac{3}{4}$$

We have shown that the basis step is true ~~for~~.

Second, we show that the induction step is true.

Assume that $P(m)$ is true, (when $n=m$).

$$\prod_{k=2}^{m}\left(1-\frac{1}{k^2}\right) = \frac{m+1}{2m}$$

Now, we need to show that $P(m+1)$ is true, given that the above is true.

$$P(m+1) = \prod_{k=2}^{m+1}\left(1-\frac{1}{k^2}\right) = P(m)\cdot$$

$$= \frac{m+1}{2m}\cdot\left(1-\frac{1}{(m+1)^2}\right) = \frac{m+1}{2m}\cdot\frac{(m+1)^2-1}{(m+1)^2}$$

$$= \frac{\cancel{m+1}}{2\cancel{m}}\cdot\frac{m^2+2\cancel{m}}{(m+1)^{\cancel{2}}} = \frac{m+2}{2(m+1)}$$

$$= \frac{(m+1)+1}{2(m+1)}$$

Thus, we show that, $P(m+1)$ is true. Therefore, by Induction, we have shown that

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n} \text{ for all } n \in \mathbb{Z}^+,\ n \geq 2$$

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3. Prove by induction that $\sum_{k=1}^{n} k\cdot k! = (n+1)!-1$, whenever n is a positive integer.

proof: First, we show that the basis is true. When $n=1$,

$$\sum_{k=1}^{1} k\cdot k! = 1\cdot 1! = 1$$

$$(n+1)!-1 = (1+1)!-1 = 1$$

We have shown that the basis step is true.

Second, we show that the induction step is true.

Assume that $P(m)$ is true;

$$\sum_{k=1}^{m} k\cdot k! = (m+1)!-1$$

Now, we need to show that $P(m+1)$ is true given that the above is true.

$$\sum_{k=1}^{m+1} k\cdot k! = \sum_{k=1}^{m} k\cdot k! + (m+1)(m+1)!$$

$$= (m+1)!-1+(m+1)!\,(m+1)$$

$$= (m+1)!\,[m+1+1]-1$$

$$= (m+2)(m+1)!-1$$

$$= [(m+1)+1]!-1$$

Thus, we have shown that $P(m+1)$ is true. Therefore, by the Induction, we have shown that

$$\sum_{k=1}^{n} k\cdot k! = (n+1)-1$$

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4. Prove by induction that $3^n < n!$ whenever n is a positive integer greater than 6.

Proof: First, we show that the basis is true. (when $n=7$)

$$3^7 = 2187$$
$$7! = 5040$$
$$\therefore 3^7 < 7!$$

$\therefore$ we have shown the basis step.

Second, we show that the induction step is true.

Assume that $P(k)$ is true. i.e. $3^k < k!$

Now, we need to show that $P(k+1)$ is true, given that the above is true.

$$3^{k+1} = 3 \cdot 3^k < 3 \cdot k! < (k+1) \cdot k! = (k+1)!$$

So. $3^{k+1} < (k+1)!$

Thus, we have shown that $P(k+1)$ is true, Therefore, by Induction, we have shown $3^n < n!$ for $n \geq 6$

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Strong Induction:

1. Let $T(n)$ be defined by

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 + \sum_{i=0}^{n-1} T(i) & \text{if } n > 0 \end{cases}$$

Then, $T(n) = 2^n$

Proof: (i) Base Case: $T(0) = 1 = 2^0$

(ii) Induction hypothesis:

Assume that $T(i) = 2^i$ for all $0 \leq i \leq (n-1)$

Then,
$$\begin{aligned} T(n) &= 1 + \sum_{i=0}^{n-1} T(i) && \text{(Definition)} \\ &= 1 + \sum_{i=0}^{n-1} 2^i && \text{(Strong Induction Hypothesis)} \\ &= 1 + (2^n - 1) \\ &= 2^n. \end{aligned}$$

So we have shown that

$T(n) = 2^n$.

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Strong Induction:

2. Consider the sequence $a_1, a_2, a_3, \cdots$ defined as:

$$a_1 = 1,\ a_2 = 2,\ a_3 = 3, \text{ and } a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

Show that $a_n < 2^n$. for $n \geq 4$

Proof: Basis: $a_4 = a_1 + a_2 + a_3$

$= 1 + 2 + 3 = 6$

$2^4 = 16$

$\therefore\ a_4 < 2^4$

Induction:

Assume that $a_i < 2^i$ for $i = 5, 6, 7, \cdots, (k-1)$.

Then $a_k = a_{k-1} + a_{k-2} + a_{k-3}$

by applying the inductive hypothesis:

$a_{k-1} < 2^{k-1}$

$a_{k-2} < 2^{k-2}$

$a_{k-3} < 2^{k-3}$

we have
$$\begin{aligned} a_k &= a_{k-1} + a_{k-2} + a_{k-3} \\ &< 2^{k-1} + 2^{k-2} + 2^{k-3} \\ &= 2^{k-3}(2^2 + 2 + 1) \\ &= 7 \cdot 2^{k-3} \\ &< 8 \cdot 2^{k-3} \\ &= 2^3 \cdot 2^{k-3} \\ &= 2^k \end{aligned}$$

So. $a_k < 2^k$

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