Recitation 12.

Section 3.2 & 2.4

P191

3.2:9: Show that $x^2+4x+17$ is $O(x^3)$ but x^3 is not $O(x^2+4x+17)$.

proof: ① We first proof $x^2+4x+17$ is $O(x^3)$

We know that $x^2+4x+17 < x^3+x^3+x^3$ for $x_0 \geq 3$

so: $x^2+4x+17 < 3x^3$

thus for all sufficiently large x

$x^2+4x+17 < Cx^3$ [$C=3$, $x_0=3$]

which implies that $x^2+4x+17$ is $O(x^3)$

② Next, we proof that x^3 is not $O(x^2+4x+17)$.

We proof it by contradiction.

We assume that x^3 is $O(x^2+4x+17)$.

then $x^3 \leq C(x^2+4x+17) \leq C(3x^2)$

$\Rightarrow$ $x^3 \leq 3Cx^2$

for $x \neq 0$ $\Rightarrow$ $x \leq 3C$ for some C and all sufficiently large x.

which is impossible! then we see a contradiction.

Therefore, x^3 is not $O(x^2+4x+17)$.

① For each of the following functions, give and prove a tight asymptotic inclusion of the form $f(n) \in \Delta(g(n))$ Where Δ is one of O, Ω, Θ.

(a) $f(n) = \log n \quad g(n) = (n+\log n)(n-1)$

proof: $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

$= \lim_{n\to\infty} \frac{\log n}{(n+\log n)(n-1)}$

$= \lim_{n\to\infty} \frac{\log n}{n^2 + n\log n - n - \log n}$

$= \lim_{n\to\infty} \frac{\log n}{n^2 + (n-1)\log n - n}$

$= \lim_{n\to\infty} \frac{1}{\frac{n^2}{\log n} + (n-1) - \frac{n}{\log n}}$

$= 0$

$\therefore f(n) \in O(g(n))$

(b) $f(n) = 2^n$, $g(n) = 2^{\log n}$

proof: $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

$= \lim_{n\to\infty} \frac{2^n}{2^{\log n}}$.

$= \lim_{n\to\infty} 2^{(n-\log n)}$

$= \infty$

$\therefore f(n) \in \Omega(g(n))$

(c) Prove that for any real numbers, $\alpha . \beta$. such that $\alpha > \beta > 0$. $\beta^n \in O(\alpha^n)$

proof: $\lim_{n\to\infty} \frac{\beta^n}{\alpha^n}$

$= \lim_{n\to\infty} (\frac{\beta}{\alpha})^n$

$\because \alpha > \beta \quad \therefore 0 < \frac{\beta}{\alpha} < 1$

$\therefore \lim_{n\to\infty} (\frac{\beta}{\alpha})^n = 0$

$\therefore \beta^n \in O(\alpha^n)$

② For the following functions, prove a tight inclusion of the form $f \in \Delta(g)$, where Δ is one of O, Ω. ④

(a) $f(n) = n^2 \log n \qquad g(n) = \left(\sum_{i=1}^{n} i\right) - \frac{n-1}{2}$

Solution: $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

$$= \lim_{n\to\infty} \frac{n^2 \log n}{\frac{n^2+1}{2}}$$

Via L'Hôpital's law and the product rule, we have.

$$\lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

$$= \lim_{n\to\infty} \frac{2n\log n + \frac{n^2}{n\ln 2}}{n}$$

$$= \lim_{n\to\infty} \frac{2n\log n + \frac{n}{\ln 2}}{n}$$

$$= \lim_{n\to\infty} 2\log n + \frac{1}{\ln 2}$$

$$= \infty$$

So. $f(n) \in \Omega(g(n))$.

(b) $f(n) = \log^2 n$, $g(n) = 4n$

Solution: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log^2 n}{4n}$

$$= \lim_{n\to\infty} \frac{2\log n \cdot \frac{1}{n \ln 2}}{4}$$ [L'Hôptial's, product. rule]

$$= \lim_{n\to\infty} \frac{\log n}{2n \ln 2}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{2 \ln 2}$$ [L'Hôptial's law]

$$= \lim_{n\to\infty} \frac{1}{2 \ln^2 2 \, n}$$

$$= 0$$

$\therefore f(n) \in O(g(n))$

(c) $f(n) = n^2$, $g(n) = n \log(8^n)$

Solution: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2}{n \log 8^n}$

$$= \lim_{n\to\infty} \frac{n^2}{n^2 \log 8}$$ [divide by n^2]

$$= \lim_{n\to\infty} \frac{1}{\log 8}$$

$$= \frac{1}{3}$$

So. $f(n) \in \Theta(g(n))$

(3) For the next 4 questions, let

$$f(n) = n^2$$
$$g(n) = \log^2(n)$$
$$h(n) = 2^n$$

(a) Give a tight asympotic charactorization of gof

solution: $g \circ f = g(f(n)) = \log^2(n^2)$

$$= [2\log n]^2$$
$$= 4\log^2 n$$
$$\in \Theta(\log^2 n)$$

(b) Give a tight asympototic characterization of goh

Solution:

$$g \circ h = g(h(n)) = \log^2(2^n)$$
$$= [\log 2^n]^2$$
$$= [n]^2$$
$$\in \Theta(n^2)$$

(c) Give a tight asymptotic charactorization of fog

Solution: $f \circ g = f(g(n)) = (\log^2 n)^2$

$$= \log^4 n$$
$$\in \Theta(\log^4 n)$$

(d) Give a tight asymptotic characterization of hog

Solution: $h \circ g = h(g(n)) = 2^{\log^2 n} = n^{\log n}$

$$\in \Theta(n^{\log n})$$

(4) For the next 3 questions, let

$$f(n) \in \Theta(n^2)$$
$$g(n) \in \Omega(n^3)$$
$$h(n) \in \Theta(n \log n)$$

Answer the following questions:

(a) Can we say $f \circ g \in \Theta(n^6)$?

Solution:

No, As a counter example: Let $g(n) = n^4$ and so $g(n) \in \Omega(n^3)$. Let $f(n) = n^2$.

Then $f \circ g = f(g(n)) = n^8 \notin \Theta(n^6)$

(b) Can we say that $f \circ f \in \Omega(n)$?

Solution:

Yes, The lower bound on each function is Linear, and so the composition will not change a linear lower bound.

(c) Can we say that $g(n) \cdot h(n) \in O(n^4 \log n)$?

Solution:

No, Let $g(n) = n^4$ and $h(n) = n \log n$ then

$g(n) \cdot h(n) = n^5 \log n \notin O(n^4 \log n)$

P192.

3.2 : 35. Suppose that $f(x)$ is $O(g(x))$ where f and g are increasing and unbounded functions. Show that $\log|f(x)|$ is $O(\log|g(x)|)$.

proof: Because $f(x)$ and $g(x)$ are increasing and unbounded, we can assume $f(x) \ge 1$ and $g(x) \ge 1$ for sufficiently large x.

$\because$ $f(x)$ is $O(g(x))$

Thus, there are constants C and k, such that

$$f(x) \le C\, g(x) \quad \text{for } x > k.$$

This implies that

$$\begin{aligned} \log f(x) &\le \log [C\, g(x)] \\ &= \log C + \log g(x) \\ &< 2 \log g(x) \quad \text{for large } x. \\ & \qquad [C = 2] \end{aligned}$$

Hence, $\log f(x)$ is $O(\log g(x))$

$\boxed{\text{Exe. 36}}$ practice after class.

P192. 3.2: 51. (b) show that $x \log x$ is $o(x^2)$

proof: $\lim_{x\to\infty} \frac{x\log x}{x^2}$

$= \lim_{x\to\infty} \frac{\log x}{x}$

using L'Hôpital's rule

$= \lim_{x\to\infty} \frac{\log x'}{x'}$

$= \lim_{x\to\infty} \frac{1}{x\ln 2}$

$= 0$

Then, $x \log x$ is $o(x^2)$

3.2 : 51 (d) show that x^2+x+1 is not $O(x^2)$.

proof: $\lim_{n\to\infty} \frac{x^2+x+1}{x^2}$

$= \lim_{n\to\infty} \left(1+\frac{1}{x}+\frac{1}{x^2}\right)$

$= 1$

$\neq 0$

Thus, x^2+x+1 is not $o(x^2)$.

P162.

2.4:19: Show that $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$ where $a_0 \cdots a_n$ is a sequence of real numbers.

Proof: Let's look at $\sum_{j=1}^{n} (a_j - a_{j-1})$

$$= \cancel{a_1} - a_0 + \cancel{a_2} - \cancel{a_1} + \cancel{a_3} - \cancel{a_2} + \cdots + a_n - \cancel{a_{n-1}}$$

$$= a_n - a_0$$

so. $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$

Exe: Determine which of the following series converge and which diverge, Justify your answer.

a) $\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$

Solution: ① The series converge.

$$\sum_{n=1}^{\infty} \frac{1}{n(n-1)} = (1-\tfrac{1}{2}) + (\tfrac{1}{2}-\tfrac{1}{3}) + \cdots + (\tfrac{1}{n-1} - \tfrac{1}{n}) + \cdots$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$$

$$= 1$$

b) $\sum_{n=1}^{\infty} 3^n e^{-n}$

Solution: $\sum_{n=1}^{\infty} 3^n e^{-n}$

$$= \sum_{n=1}^{\infty} \left(\frac{3}{e}\right)^n.$$

because $\frac{3}{e} > 1$, $\therefore \left(\frac{3}{e}\right)^n$ diverges to ∞

c) $\sum_{n=1}^{\infty} \frac{n}{10n+17}$

Solution: $a_n = \frac{n}{10n+17}$ in this series.

Let's look at the limit of this sequence.

$$\lim_{n\to\infty} \frac{n}{10n+17} = \lim_{n\to\infty} \frac{1}{10+\frac{17}{n}} = \frac{1}{10}$$

Since the limit of the sequence does Not equal zero, the n^{th} term test for divergence tells us that the series must diverge.