3. List the ordered pairs in the relations on \( \{1, 2, 3\} \) corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) \[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]
\( (1, 1), (1, 2), (2, 2), (3, 1), (3, 3) \)

b) \[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]
\( (1, 2), (2, 2), (3, 2) \)

c) \[
\begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}
\]
\( (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3) \)
13. Let $R$ be the relation represented by the matrix $M_R$:

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Find the matrix representing:

a) $R^1$

Sol: $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

b) $R^\top$

Sol: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

c) $R^2$

Sol: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

3. Let $R$ be the relation $\{ (a, b) | a \text{ divides } b \}$ on the set of integers. What is $R \cup R^\top$?

Solution: $R \cup R^\top = \{ (a, b) | a \text{ divides } b \text{ or } b \text{ divides } a \}$
8.1.3: Let \( S = \{1, 2, 3, 4\} \). Determine whether each relation \( R \) on \( S \) has the listed properties.

\[
R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}
\]

**Question 1:** Draw the 0-1 matrix representation of \( R \).

**Solution:**

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**Question 2:** Draw the digraph representation of \( R \).

**Solution:**

\[
\begin{array}{cc}
1 & \quad 4 \\
2 & \quad 3
\end{array}
\]

3. Is \( R \) symmetric?
   
   Yes

4. Is \( R \) transitive?
   
   Yes

5. Is \( R \) reflexive?
   
   Yes
8.9. Suppose that \( A \) is a nonempty set, and \( f \) is a function that has \( A \) as its domain. Let \( R \) be the relation on \( A \) consisting of all ordered pairs \((x,y)\) such that \( f(x) = f(y) \).

a) Show that \( R \) is an equivalence relation on \( A \).

solution: \((x,x) \in R\) because \( f(x) = f(x) \).

Hence, \( R \) is reflexive.

\((x,y) \in R\) iff \( f(x) = f(y) \), which holds iff \( f(y) = f(x) \) if \((y, x) \in R\).

Hence, \( R \) is symmetric.

If \((x,y) \in R\), and \((y,z) \in R\), then \( f(x) = f(y) \) and \( f(y) = f(z) \). Hence, \( f(x) = f(z) \). Thus, \((x,z) \in R\).

It follows that \( R \) is transitive.

b) What are the equivalence classes of \( R \)?

the sets \( f^{-1}(b) \) for \( b \) in the range of \( f \).
Which of these relations on the set of all functions from \( \mathbb{Z} \) to \( \mathbb{Z} \) are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) \[ \{(f, g) \mid f(1) = g(1)\} \]

Solution: It is reflexive. Transitive.

- if we have \( f \mathrel{R} g \), i.e. \( f(1) = g(1) \)
  and \( g \mathrel{R} h \), i.e. \( g(1) = h(1) \)
  then we will have \( f(1) = g(1) = h(1) \Rightarrow f(1) = h(1) \)
  then we have \( f \mathrel{R} h \).

- It is reflexive.

- if we have \( f \mathrel{R} g \), i.e. \( f(1) = g(1) \)
  then we will have \( g(1) = f(1) \), which implies \( g \mathrel{R} f \).

So, if \( f \mathrel{R} g \), then \( g \mathrel{R} f \). It is symmetric.

Since \( R \) has all three properties, it is an equivalence relation.
\( b \) \quad \{ (f, g) \mid f(0) = g(0), \text{ or } f(1) = g(1) \}.

Solution: \( R \) is reflexive:

\[ \because \text{ we have } f(0) = f(0) \text{ or } f(1) = f(1) \quad \Rightarrow \quad R \text{ is reflexive.} \]

\[ \rightarrow \text{ It is symmetric.} \]

\[ \because \text{ if we have } \ f \ R \ g \text{, i.e., } f(0) = g(0) \text{ or } f(1) = g(1) \]

\[ \text{then we have } \ g \ R \ f \text{, i.e., } g(0) = f(0) \text{ or } g(1) = f(1). \]

\[ \therefore \text{ } R \text{ is symmetric.} \]

\[ \rightarrow \text{ It is not transitive.} \]

\[ \because \text{ if we have } \ f \ R \ g \text{, i.e., } f(0) = g(0) \text{ or } f(1) = g(1) \]

\[ \text{and } \ g \ R \ h \text{, i.e., } g(0) = h(0) \text{ or } g(1) = h(1) \]

\[ \therefore \text{ Counter example is,} \]

\[ \text{when } \ f(0) = g(0) \wedge f(1) = g(1) \text{ i.e., } f \ R \ g \]

\[ g(0) = h(0) \wedge g(1) = h(1) \text{ i.e., } g \ R \ h. \]

\[ \therefore \text{ we have } f(0) = h(0) \wedge f(1) = h(1) \]

\[ \therefore \text{ Then, } f \ R \ h. \]

\[ \therefore \text{ it is not transitive.} \]

\[ \rightarrow \text{ Since } R \text{ does not have all three properties,} \]

\[ \therefore \text{ it is not an equivalence relation.} \]
\[ (f, g) \mid f(x) - g(x) \neq 0 \text{ for all } x \in \mathbb{R} \]

(c) \[ (f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbb{R} \]

Solution: \[ \rightarrow \text{ It is not reflexive.} \]

- If we have \[ f R f \text{, then we should have} \]
  \[ f(x) - f(x) = 1 \text{ which is impossible.} \]
  \[ \text{So, } R \text{ is not reflexive, which means } f R f. \]

\[ \rightarrow \text{ It is not symmetric.} \]

- If we have \[ f R g \text{, i.e. } f(x) - g(x) = 1 \]
  then, \[ g(x) - f(x) = -1 \text{, which means } g \not\sim f \]

\[ \rightarrow \text{ It is not transitive.} \]

- If we have \[ f R g \text{, i.e. } f(x) - g(x) = 1 \]
  and we have \[ g R h \text{, i.e. } g(x) - h(x) = 1 \]
  \[ \Rightarrow f(x) - h(x) = 2 \]

\[ \rightarrow \text{ It is not transitive.} \]

Since \[ R \text{ does not have all three properties,} \]
then \[ R \text{ is not an equivalence relation.} \]
P5.28 32. \( \varepsilon \subseteq \%

* For the following relations on the set of real numbers:

\[ R_1 = \{(a, b) \in \mathbb{R}^2 \mid a > b\}, \text{ the "greater than" relation,} \]
\[ R_2 = \{(a, b) \in \mathbb{R}^2 \mid a \geq b\}, \text{ the "greater than or equal to" relation,} \]
\[ R_3 = \{(a, b) \in \mathbb{R}^2 \mid a < b\}, \text{ the "less than" relation,} \]
\[ R_4 = \{(a, b) \in \mathbb{R}^2 \mid a \leq b\}, \text{ the "less than or equal to" relation,} \]
\[ R_5 = \{(a, b) \in \mathbb{R}^2 \mid a = b\}, \text{ the "equal to" relation,} \]
\[ R_6 = \{(a, b) \in \mathbb{R}^2 \mid a \neq b\}, \text{ the "unequal to" relation.} \]

Find:

\( a) R_2 \cup R_4 \]
= \( R^2 \)

\( b) R_3 \cup R_6 \]
= \( R_6 \quad (\because R_6 \text{ includes all relations in } R_3) \)

\( c) R_3 \cap R_6 \]
= \( R_3 \)

\( d) R_4 \cap R_6 \]
= \( \{(a, b) \in \mathbb{R}^2 \mid a \leq b\} \cap \{(a, b) \in \mathbb{R}^2 \mid a \neq b\} \]
= \( \{(a, b) \in \mathbb{R}^2 \mid a < b\} \]
= \( R_3 \)

\( e) R_3 - R_6 \]
= \( \{(a, b) \in \mathbb{R}^2 \mid a < b\} - \{(a, b) \in \mathbb{R}^2 \mid a \neq b\} \]
= \( \emptyset \)
(g) \[ R_6 - R_3 \]
\[ = \{ (a, b) \in \mathbb{R}^2 \mid a + b \leq 1 \} - \{ (a, b) \in \mathbb{R}^2 \mid a + b \geq 2 \} \]
\[ = \{ (a, b) \in \mathbb{R}^2 \mid a > b \} \]
\[ = R_1 \]

(g) \[ R_2 \oplus R_6 \]
\[ = \{ (a, b) \in \mathbb{R}^2 \mid a \geq b \} \oplus \{ (a, b) \in \mathbb{R}^2 \mid a + b \leq 1 \} \]
\[ = \{ (a, b) \in \mathbb{R}^2 \mid a < b \} \]
\[ = R_4 \]

(h) \[ R_3 \oplus R_5 \]
\[ = \{ (a, b) \in \mathbb{R}^2 \mid a < b \} \oplus \{ (a, b) \in \mathbb{R}^2 \mid a = b \} \]
\[ = \{ (a, b) \in \mathbb{R}^2 \mid a < b \} \]
\[ = R_4 \]