review session:

— Relations
— Partial orders
— Algorithm
— Asymptotics
— Sequence and Summations
— Induction.
— Recurison
— Master Theorem
— Combinatorics:

→ the materials before the Midterm.

Relations.

Section 8.1. 8.3, 8.4. 8.5.

→ Relation:

Definition, representation. relation on a set.

→ Properties:

Reflexivity, symmetry, antisymmetric, irreflexive, asymmetric.

→ Combing relations:

$\cup$. $\cap$. $-$. $\circ$

→ Representing relations.

matrices, directed graph.

→ Closure. of relations.

→ Equivalence relation:

Relations:

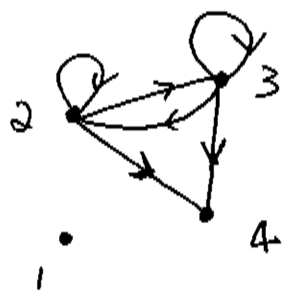
* Consider the relation R_1, R_2 on the set $S = \{1, 2, 3, 4\}$ defined as follows:

$$R_1 = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$$
$$R_2 = \{(1,2), (2,1), (2,2), (3,3)\}$$

a) Draw the 0-1 matrix representation of R_1

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b) Draw the diagraph representation of R_1



c) Compute the matrix for $R_1 \cup R_2$.

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

d) Is R_1 reflexive ?

No, since $(1,1) \notin R$.

e) Is R_1 symmetric ?

No. Since $(2,4) \in R$. but $(4,2) \notin R$

f) Is R_1 transitive ?

Yes. it is transitive.

g) Is R_1 an equivalent relation ?

No. Since equivalence relations must be reflexive, symmetric, and transitive, but R_1 is not reflexive nor symmetric.

Partial order

Section 8.6.

→ Definition

→ Principle of well-ordered induction.

→ Lexicographic orderings

→ Hasse Diagrams

→ Extremal elements.

→ Lattices

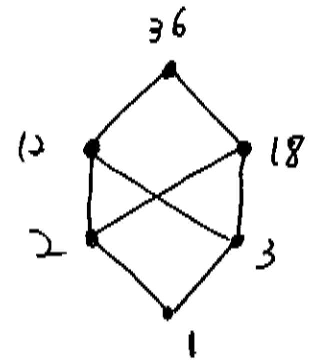
→ Topological Sorting.

Partial order:

* Answer these questions from the poset:

({1, 2, 3, 12, 18, 36}, |) i.e. a R b if a divides b

a) Draw the Hasse diagram for this poset:



b) what's the maximal elements in this poset?

{36}

c) what's the minimal elements in this poset?

{1}

d) what's the greatest lower bound of {2. 3}?

{1}

e) what's the upper bound of {2. 3}?

{12. 18. 36}

f) what's the least upper bounds of {2. 3}?

NONE.

e) Is this poset a Lattice?

No, ∵ for {12. 18. 36}, none of them precedes the other two w.r.t. the ordering of this poset.

Algorithm and Asympototics.

Section 3.3 . 3.2.

* For each of the following functions, give and prove a tight asymptotic inclusion of the form $f(n) \in \Delta(g(n))$ where Δ is one of O, Ω, Θ

(a) $f(n) = \log n$ $\quad g(n) = (n + \log n)(n+1)$

Solution: $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

$$= \lim_{n\to\infty} \frac{\log n}{(n+\log n)(n+1)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{\left(1+\frac{1}{n\ln 2}\right)(n+1) + (n+\log n)}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\ln 2}}{n^2+n+\frac{n}{\ln 2}+\frac{1}{\ln 2} + n^2 + n\log n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\ln 2}}{2n^2 + \left(1+\frac{1}{\ln 2}\right) n + n\log n + \frac{1}{\ln 2}}$$

$$= 0$$

$\therefore f(n) \in O(g(n))$

Sequences and Summations

Section 2.4.

* Determine which of the following series converge and which diverse. Justify your answer.

a) $\sum_{n=1}^{\infty} 3^n e^{-n}$

solution: $\sum_{n=1}^{\infty} 3^n e^{-n}$

$$= \sum_{n=1}^{\infty} \left(\frac{3}{e}\right)^n$$

$\because$ ~~$\Rightarrow$~~ $3 > e$ $\therefore$ $\frac{3}{e} > 1$

$\Rightarrow$ $\left(\frac{3}{e}\right)^n \to \infty$ when $n \to \infty$

so. it diverges.

b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$$

so when $n \to \infty$. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \to 1$

Induction:

Section: 4.1. and 4.2.

* Prove by induction that all numbers of the form $11^n - 4^n$ are divisible by 7. for all $n \geq 1$

Proof: Basic step:

when $n=1$: $11^1 - 4^1 = 11 - 4 = 7$ which is divisible by 7. so, Basic step is true.

Now we assume that when $n=k$, $11^k - 4^k$ is divisible by 7.

Then: Let's check when $n=k+1$,

when $n=k+1$,

$$11^{k+1} - 4^{k+1} = 11 \cdot (11^k) - 4 \cdot (4^k)$$

$$= 11(11^k - 4^k) - 4 \cdot 4^k + 11 \cdot 4^k$$

$$= 11(11^k - 4^k) + 4^k(11-4)$$

$$= 11 \cdot (11^k - 4^k) + 7 \cdot 4^k$$

$\because$ by assumption, $11^k - 4^k$ is divisible by 7,

thus. $11 \cdot (11^k - 4^k) + 7 \cdot 4^k$ is divisible by 7

Therefore, by PMI, we have shown that all numbers of the form $11^n - 4^n$ are divisible by 7 for all $n \geq 1$

Recursion:

Section 7.1. and 7.2.

* Let $S_0=1$, $S_1=4$, and $S_n = 5S_{n-1} - 6S_{n-2}$ for $n \geq 2$

a) Compute the value of S_3.

Solution: $S_3 = 5S_2 - 6S_1$

$$= 5(5S_1 - 6S_0) - 6S_1$$
$$= 5[5\times4 - 6\times1] - 6\times4$$
$$= 5\times14 - 24$$
$$= 46$$

b) Using the theorem on recursion relations given in class, derive an explicit formular for S_n.

Solution: The characteristic equation that we need to solve is:

$$r^2 - 5r + 6 = 0$$
$$\Rightarrow r_1 = 2 \quad r_2 = 3$$

So, we express $S_n = 5S_{n-1} - 6S_{n-2}$ in terms of

$$S_n = \alpha_1 2^n + \alpha_2 3^n$$

$$\because \begin{cases} S_0 = 1 \\ S_1 = 4 \end{cases} \Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 1 \\ 2\alpha_1 + 3\alpha_2 = 4 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -1 \\ \alpha_2 = 2. \end{cases}$$

Thus: $S_n = -2^n + 2\cdot3^n.$

Master Theorem

Section 7.3

1. $T(n) = 3T(n/2) + n^2$

Solution: $\Rightarrow \begin{cases} a = 3 \\ b = 2 \\ c = 1 \\ d = 2 \end{cases}$

$\therefore\ a = 3 \quad b^d = 2^2 = 4$

$\rightarrow\ a < b^d$

Then, $T(n) \in \Theta(n^d) = \Theta(n^2)$

2. $T(n) = 3T(n/2) + n\log n$

Solution: $\begin{cases} a = 3 \\ b = 2 \end{cases}$ but $f(n)$ is not polynomial,

However, $f(n) = n\log n$, $f(n) \in \Theta(n\log n)$, so $k = 1$

$T(n) \in \Theta(n^{\log_2^3} \log^{k+1} n)$

$\in \Theta(n^{\log_2^3} \log^2 n)$

3. $T(n) = 0.5\,T(n/2) + n$

Solution: Theorem dose not apply,

Since $a = 0.5 < 1$

4. $T(n) = 64\,T(n/8) - n^2 \log n$.

Solution: $\begin{cases} a = 64 \\ b = 8 \\ c = -1 \\ d = 2 \end{cases}$

So Theorem does not apply ($c < 0$).

5. $T(n) = T(n/2) + 2n - \cos n$

Solution: $\begin{cases} a = 1 \\ b = 2 \end{cases}$

but $f(n)$ is not monotone

Thus, Theorem does not hold.

6. $T(n) = 2T(n/4) + n^{0.51}$

Solution: $\begin{cases} a = 2 \\ b = 4 \\ c = 1 \\ d = 0.51 \end{cases}$

$\because$ $a = 2$. $b^d = 4^{0.51}$

$\Rightarrow$ $a < b^d$

Thus. $T(n) \in \Theta(n^{0.51})$

Combinatonics:

Section 5.1 ~ 6.6 , 7.5 - 7.6

* Suppose that you aro the coach of your local basketball camp for kids. There are 25 kids in the camp with the following distribution.

Age.	Boys	Girls.	Total.
8 years old	3	4	7
9	5	4	9
10	8	1	9
Total	16	9	25.

a) How many ways are there to form a basketball team of 5 players?

Solution:

$\because$ There are 25 players in total, we choose 5.

and get: $C_{25}^{5} = \frac{25!}{5!\,(25-5)!} = \frac{25\cdot 24\cdot 23\cdot 22\cdot 21\cdot \cancel{20!}}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot \cancel{20!}}$

$=$

b) How many ways are there to form a basketball team of 3 boys and 2 girls?

Solution: There are 16 boys and 9 girls.

so there are C_{16}^{3} ways to choose 3 boys.

there are C_{9}^{2} ways to choose 2 girls.

Since it is a sequence of event, we multiply them, and get

$C_{16}^{3} \cdot C_{9}^{2} = \cancel{560}\ 560 \cdot 36 = 20160$

(c) How many ways are there to form a basketball team of 5 players with at least one boy?

Solution:

Let's find out how many ways are there to form a team with all girls, That is

$$C_9^5 = \frac{9!}{5!\,(9-5)!} = 126$$

So, the total number of ways to form a team with at least one boy is then the total number of ways minus C_9^5.

which is: $C_{25}^5 - C_9^5 = 53130 - 126 = 53004$