Sequences & Summations

Though you should be (at least intuitively) familiar with sequences and summations, we give a quick review.

**Sequences**

**Definition**

A sequence is a function from a subset of integers to a set $S$. We use the notation(s):\

$\{a_n\}, \{a_n\}_{n=1}^{\infty}, \{a_n\}_{n=0}^{\infty}, \{a_n\}_{n=0}^{\infty}$

Each $a_n$ is called the $n$-th term of the sequence.

We rely on context to distinguish between a sequence and a set; though there is a connection.
Consider the sequence \( \{ \left( 1 + \frac{1}{n} \right)^n \}_{n=1}^{\infty} \).

The terms are:

- \( a_1 = (1 + 1)^1 = 2 \)
- \( a_2 = (1 + \frac{1}{2})^2 = 2.25000 \)
- \( a_3 = (1 + \frac{1}{3})^3 = 2.37037 \)
- \( a_4 = (1 + \frac{1}{4})^4 = 2.44140 \)
- \( a_5 = (1 + \frac{1}{5})^5 = 2.48832 \)

What is this sequence?

The sequence corresponds to \( e \):

\[
\lim_{n \to \infty} \left\{ \left( 1 + \frac{1}{n} \right)^n \right\} = e = 2.71828 \ldots
\]

The sequence \( \{ h_n \}_{n=1}^{\infty} = \frac{1}{n} \) is known as the harmonic sequence. The sequence is simply

\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]

This sequence is particularly interesting because its summation is divergent:

\[
\sum_{n=1}^{\infty} \frac{1}{n} = \infty
\]
**Proportions I**

**Definition**

A geometric progression is a sequence of the form

\[ a, ar, ar^2, ar^3, \ldots, ar^n, \ldots \]

Where \( a \in \mathbb{R} \) is called the *initial term* and \( r \in \mathbb{R} \) is the *common ratio*.

A geometric progression is a discrete analogue of the exponential function

\[ f(x) = ar^x \]

**Proportions II**

**Definition**

An arithmetic progression is a sequence of the form

\[ a, a + d, a + 2d, a + 3d, \ldots, a + nd, \ldots \]

Where \( a \in \mathbb{R} \) is called the *initial term* and \( r \in \mathbb{R} \) is the *common difference*.

Again, an arithmetic progression is a discrete analogue of the linear function,

\[ f(x) = dx + a \]

**Proportions III**

**Example**

A common geometric progression in computer science is

\( \{a_n\} = \frac{1}{2^n} \)

Here, \( a = 1 \) and \( r = \frac{1}{2} \)

Table 1 on Page 153 (Rosen) has useful sequences.
Summations I

You should be very familiar with Summation notation:

\[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_{n-1} + a_n \]

Here, \( j \) is the index of summation, \( m \) is the lower limit, and \( n \) is the upper limit.

Often times, it is useful to change the lower/upper limits; which can be done in a straightforward manner (though we must be careful).

\[ \sum_{j=1}^{n} a_j = \sum_{j=0}^{n-1} a_{j+1} \]

Sometimes we can express a summation in closed form. Geometric series, for example:

Summations II

Theorem

For \( a, r \in \mathbb{R}, r \neq 0 \),

\[ \sum_{i=0}^{n} ar^i = \begin{cases} ar^{n+1}-a & \text{if } r \neq 1 \\ \frac{r^{n+1}-1}{r-1}a & \text{if } r = 1 \end{cases} \]

Summations III

Double summations often arise when analyzing an algorithm.

\[ \sum_{i=1}^{n} \sum_{j=1}^{i} a_j = a_1 + a_1 + a_2 + a_1 + a_2 + a_3 + \cdots + a_1 + a_2 + a_3 + \cdots + a_n \]

Summations can also be indexed over elements in a set.

\[ \sum_{s \in S} f(s) \]

Table 2 on Page 157 (Rosen) has useful summations.
Series

When we take the sum of a sequence, we get a series. We’ve already seen a closed form for geometric series.

Some other useful closed forms include the following.

\[ \sum_{i=1}^{u} 1 = u - l + 1, \text{ for } l \leq u \]
\[ \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \]
\[ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]
\[ \sum_{i=0}^{n} k^\alpha \approx \frac{1}{k+1} \]
\[ \sum_{i=1}^{n} \log i \approx n \log n \]

Notes

Infinite Series I

Though we will mostly deal with finite series (i.e. an upper limit of \( n \) for a fixed integer), infinite series are also useful.

Example

Infinite Series II

Consider the geometric series
\[ \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots \]
This series converges to 2. However, the geometric series
\[ \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots \]
does not converge. However, note that \( \sum_{n=0}^{\infty} 2^n = 2^{n+1} - 1 \)

In fact, we can generalize this as follows.

Lemma
Infinite Series III

A geometric series converges if and only if the absolute value of the common ratio is less than 1.