

Relations

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Computer Science & Engineering 235
Introduction to Discrete Mathematics
Sections 8.1, 8.3–8.5 of Rosen

Recall that a relation between elements of two sets is a subset of their Cartesian product (of ordered pairs).

Definition

A *binary relation* from a set A to a set B is a subset

$$R \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Note the difference between a relation and a function: in a relation, each $a \in A$ can map to multiple elements in B . Thus, relations are generalizations of functions.

If an ordered pair $(a, b) \in R$ then we say that a is *related* to b . We may also use the notation aRb and $aR \not b$.

To represent a relation, you can enumerate every element in R .

Example

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$ let R be a relation from A to B as follows:

$$R = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), \\ (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_5, b_1)\}$$

You can also represent this relation graphically.

Relations

CSE235

Introduction

Properties

Representations

Closures

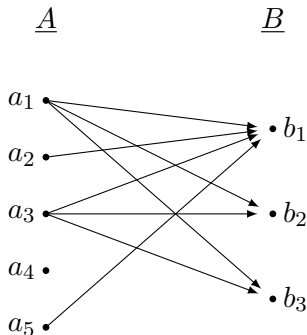
Equivalence
Relations

Figure: Graphical Representation of a Relation

Definition

A relation on the set A is a relation from A to A . I.e. a subset of $A \times A$.

Example

The following are binary relations on \mathbb{N} :

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a, b \in \mathbb{N}, \frac{a}{b} \in \mathbb{Z}\}$$

$$R_3 = \{(a, b) \mid a, b \in \mathbb{N}, a - b = 2\}$$

EXERCISE: Give some examples of ordered pairs $(a, b) \in \mathbb{N}^2$ that are not in each of these relations.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

There are several properties of relations that we will look at. If the ordered pairs (a, a) appear in a relation on a set A for every $a \in A$ then it is called reflexive.

Definition

A relation R on a set A is called *reflexive* if

$$\forall a \in A ((a, a) \in R)$$

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Recall the following relations; which is reflexive?

$$R_1 = \{(a, b) \mid a \leq b\}$$

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Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Recall the following relations; which is reflexive?

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$$R_3 = \{(a, b) \mid a, b \in \mathbb{N}, a - b = 2\}$$

- R_1 is reflexive since for every $a \in \mathbb{N}$, $a \leq a$.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Recall the following relations; which is reflexive?

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$$R_3 = \{(a, b) \mid a, b \in \mathbb{N}, a - b = 2\}$$

- R_1 is reflexive since for every $a \in \mathbb{N}$, $a \leq a$.
- R_2 is also reflexive since $\frac{a}{a} = 1$ is an integer.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Recall the following relations; which is reflexive?

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$$R_3 = \{(a, b) \mid a, b \in \mathbb{N}, a - b = 2\}$$

- R_1 is reflexive since for every $a \in \mathbb{N}$, $a \leq a$.
- R_2 is also reflexive since $\frac{a}{a} = 1$ is an integer.
- R_3 is *not* reflexive since $a - a = 0$ for every $a \in \mathbb{N}$.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Definition

A relation R on a set A is called *symmetric* if

$$(b, a) \in R \iff (a, b) \in R$$

for all $a, b \in A$.

A relation R on a set A is called *antisymmetric* if

$$\forall a, b, \left[((a, b) \in R \wedge (b, a) \in R) \rightarrow a = b \right]$$

for all $a, b \in A$.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Some things to note:

- A symmetric relationship is one in which if a is related to b then b *must* be related to a .
- An antisymmetric relationship is similar, but such relations hold only when $a = b$.
- An antisymmetric relationship is *not* a reflexive relationship.
- A relation can be both symmetric and antisymmetric or neither or have one property but not the other!
- A relation that is not symmetric is *not* necessarily *asymmetric*.

Symmetric Relations

Example

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Let $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Is R reflexive?
Symmetric? Antisymmetric?

Symmetric Relations

Example

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Let $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Is R reflexive?
Symmetric? Antisymmetric?

- It is clearly not reflexive since for example $(2, 2) \notin R$.

Symmetric Relations

Example

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Example

Let $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Is R reflexive?
Symmetric? Antisymmetric?

- It is clearly not reflexive since for example $(2, 2) \notin \mathbb{R}$.
- It is symmetric since $x^2 + y^2 = y^2 + x^2$ (i.e. addition is commutative).

Symmetric Relations

Example

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Example

Let $R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Is R reflexive?
Symmetric? Antisymmetric?

- It is clearly not reflexive since for example $(2, 2) \notin R$.
- It is symmetric since $x^2 + y^2 = y^2 + x^2$ (i.e. addition is commutative).
- It is not antisymmetric since $(\frac{1}{3}, \frac{\sqrt{8}}{3}) \in R$ and $(\frac{\sqrt{8}}{3}, \frac{1}{3}) \in R$ but $\frac{1}{3} \neq \frac{\sqrt{8}}{3}$

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Definition

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ for all $a, b, c \in R$. Equivalently,

$$\forall a, b, c \in A ((aRb \wedge bRc) \rightarrow aRc)$$

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Example

Is the relation $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$ transitive?

Example

Is the relation $R = \{(a, b), (b, a), (a, a)\}$ transitive?

Transitivity

Examples

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Example

Is the relation $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$ transitive?

Yes it is transitive since $(x \leq y) \wedge (y \leq z) \Rightarrow x \leq z$.

Example

Is the relation $R = \{(a, b), (b, a), (a, a)\}$ transitive?

Transitivity

Examples

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Example

Is the relation $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$ transitive?

Yes it is transitive since $(x \leq y) \wedge (y \leq z) \Rightarrow x \leq z$.

Example

Is the relation $R = \{(a, b), (b, a), (a, a)\}$ transitive?

No since bRa and aRb but $bR \not b$.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Is the relation

$$\{(a, b) \mid a \text{ is an ancestor of } b\}$$

transitive?

Example

Is the relation $\{(x, y) \in \mathbb{R}^2 \mid x^2 \geq y\}$ transitive?

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Example

Is the relation

$$\{(a, b) \mid a \text{ is an ancestor of } b\}$$

transitive?

Yes, if a is an ancestor of b and b is an ancestor of c then a is also an ancestor of c (who is the youngest here?).

Example

Is the relation $\{(x, y) \in \mathbb{R}^2 \mid x^2 \geq y\}$ transitive?

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Is the relation

$$\{(a, b) \mid a \text{ is an ancestor of } b\}$$

transitive?

Yes, if a is an ancestor of b and b is an ancestor of c then a is also an ancestor of c (who is the youngest here?).

Example

Is the relation $\{(x, y) \in \mathbb{R}^2 \mid x^2 \geq y\}$ transitive?

No. For example, $(2, 4) \in R$ and $(4, 10) \in R$ (i.e. $2^2 \geq 4$ and $4^2 = 16 \geq 10$) but $2^2 < 10$ thus $(2, 10) \notin R$.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Definition

- A relation is *irreflexive* if

$$\forall a [(a, a) \notin R]$$

- A relation is *asymmetric* if

$$\forall a, b [(a, b) \in R \rightarrow (b, a) \notin R]$$

Lemma

A relation R on a set A is asymmetric if and only if

- R is irreflexive and
- R is antisymmetric.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Relations are simply sets, that is subsets of ordered pairs of the Cartesian product of a set.

It therefore makes sense to use the usual set operations, intersection \cap , union \cup and set difference $A \setminus B$ to *combine* relations to create new relations.

Sometimes combining relations endows them with the properties previously discussed. For example, two relations may not be transitive alone, but their union may be.

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Let

$$A = \{1, 2, 3, 4\}$$

$$B = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$$

$$R_2 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

Then

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Let

$$A = \{1, 2, 3, 4\}$$

$$B = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$$

$$R_2 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

Then

- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2)\}$

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence

Relations

Example

Let

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$$R_1 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$$

$$R_2 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

Then

- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2)\}$
- $R_1 \cap R_2 = \{(1, 2), (1, 3)\}$

Example

Let

$$A = \{1, 2, 3, 4\}$$

$$B = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$$

$$R_2 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

Then

- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2)\}$
- $R_1 \cap R_2 = \{(1, 2), (1, 3)\}$
- $R_1 \setminus R_2 = \{(1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$

Example

Let

$$A = \{1, 2, 3, 4\}$$

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$$R_1 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$$

$$R_2 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

Then

- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2)\}$
- $R_1 \cap R_2 = \{(1, 2), (1, 3)\}$
- $R_1 \setminus R_2 = \{(1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$
- $R_2 \setminus R_1 = \{(1, 1), (2, 3)\}$

Definition

Let R_1 be a relation from the set A to B and R_2 be a relation from B to C . I.e. $R_1 \subseteq A \times B, R_2 \subseteq B \times C$. The *composite* of R_1 and R_2 is the relation consisting of ordered pairs (a, c) where $a \in A, c \in C$ and for which there exists an element $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. We denote the composite of R_1 and R_2 by

$$R_1 \circ R_2$$

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Using this *composite* way of combining relations (similar to function composition) allows us to recursively define *powers* of a relation R .

Definition

Let R be a relation on A . The powers, $R^n, n = 1, 2, 3, \dots$, are defined recursively by

$$\begin{aligned}R^1 &= R \\ R^{n+1} &= R^n \circ R\end{aligned}$$

Powers of Relations

Example

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

Consider $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$

 $R^2 =$ $R^3:$ $R^4:$

Notice that $R^n = R^3$ for $n=4, 5, 6, \dots$

Relations

CSE235

Introduction

Properties

Reflexivity

Symmetry

Transitivity

Combining
Relations

Representations

Closures

Equivalence
Relations

The powers of relations give us a nice characterization of transitivity.

Theorem

A relation R is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$

Relations

CSE235

Introduction

Properties

Representations

Matrices
Directed Graphs

Closures

Equivalence
Relations

We have seen ways of graphically representing a function/relation between two (different) sets—specifically a graph with arrows between nodes that are related.

We will look at two alternative ways of representing relations; 0-1 matrices and directed graphs.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

A 0-1 matrix is a matrix whose entries are either 0 or 1.

Let R be a relation from $A = \{a_1, a_2, \dots, a_n\}$ to $B = \{b_1, b_2, \dots, b_m\}$.

Note that we have induced an ordering on the elements in each set. Though this ordering is arbitrary, it is important to be consistent; that is, once we fix an ordering, we stick with it.

In the case that $A = B$, R is a relation *on* A , and we choose the same ordering.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

The relation R can therefore be represented by a $(n \times m)$ sized 0-1 matrix $\mathbf{M}_R = [m_{i,j}]$ as follows.

$$m_{i,j} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Intuitively, the (i, j) -th entry is 1 if and only if $a_i \in A$ is related to $b_j \in B$.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

An important note: the choice of row or column-major form is important. The (i, j) -th entry refers to the i -th *row* and j -th *column*. The size, $(n \times m)$ refers to the fact that \mathbf{M}_R has n *rows* and m *columns*.

Though the choice is arbitrary, switching between row-major and column-major is a bad idea, since for $A \neq B$, the Cartesian products $A \times B$ and $B \times A$ are not the same.

In matrix terms, the *transpose*, $(\mathbf{M}_R)^T$ does not give the same relation. This point is moot for $A = B$.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence

Relations

$$A \left\{ \begin{array}{l} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \right. \begin{array}{c} \overbrace{\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix}} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Let's take a quick look at the example from before.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence

Relations

Example

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$ let R be a relation from A to B as follows:

$$R = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_5, b_1)\}$$

What is M_R ?

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence

Relations

Example

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$ let R be a relation from A to B as follows:

$$R = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_5, b_1)\}$$

What is M_R ?

Clearly, we have a (5×3) sized matrix.

Example

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$ let R be a relation from A to B as follows:

$$R = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_5, b_1)\}$$

What is \mathbf{M}_R ?

Clearly, we have a (5×3) sized matrix.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence

Relations

A 0-1 matrix representation makes checking whether or not a relation is reflexive, symmetric and antisymmetric very easy.

Reflexivity – For R to be reflexive, $\forall a(a, a) \in R$. By the definition of the 0-1 matrix, R is reflexive if and only if $m_{i,i} = 1$ for $i = 1, 2, \dots, n$. Thus, one simply has to check the diagonal.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Symmetry – R is symmetric if and only if for all pairs (a, b) , $aRb \Rightarrow bRa$. In our defined matrix, this is equivalent to $m_{i,j} = m_{j,i}$ for every pair $i, j = 1, 2, \dots, n$.

Alternatively, R is symmetric if and only if $\mathbf{M}_R = (\mathbf{M}_R)^T$.

Antisymmetry – To check antisymmetry, you can use a disjunction; that is R is antisymmetric if $m_{i,j} = 1$ with $i \neq j$ then $m_{j,i} = 0$. Thus, for all $i, j = 1, 2, \dots, n$, $i \neq j$, $(m_{i,j} = 0) \vee (m_{j,i} = 0)$.

What is a simpler logical equivalence?

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Symmetry – R is symmetric if and only if for all pairs (a, b) , $aRb \Rightarrow bRa$. In our defined matrix, this is equivalent to $m_{i,j} = m_{j,i}$ for every pair $i, j = 1, 2, \dots, n$.

Alternatively, R is symmetric if and only if $\mathbf{M}_R = (\mathbf{M}_R)^T$.

Antisymmetry – To check antisymmetry, you can use a disjunction; that is R is antisymmetric if $m_{i,j} = 1$ with $i \neq j$ then $m_{j,i} = 0$. Thus, for all $i, j = 1, 2, \dots, n$, $i \neq j$, $(m_{i,j} = 0) \vee (m_{j,i} = 0)$.

What is a simpler logical equivalence?

$$\forall i, j = 1, 2, \dots, n; i \neq j (\neg(m_{i,j} \wedge m_{j,i}))$$

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Is R reflexive? Symmetric? Antisymmetric?

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Is R reflexive? Symmetric? Antisymmetric?

- Clearly it is not reflexive since $m_{2,2} = 0$.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Is R reflexive? Symmetric? Antisymmetric?

- Clearly it is not reflexive since $m_{2,2} = 0$.
- It is not symmetric either since $m_{2,1} \neq m_{1,2}$.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Is R reflexive? Symmetric? Antisymmetric?

- Clearly it is not reflexive since $m_{2,2} = 0$.
- It is not symmetric either since $m_{2,1} \neq m_{1,2}$.
- It is, however, antisymmetric. You can verify this for yourself.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Combining relations is also simple—union and intersection of relations is nothing more than entry-wise boolean operations.

Union – An entry in the matrix of the union of two relations $R_1 \cup R_2$ is 1 if and only if at least one of the corresponding entries in R_1 or R_2 is one. Thus

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}$$

Intersection – An entry in the matrix of the intersection of two relations $R_1 \cap R_2$ is 1 if and only if *both* of the corresponding entries in R_1 and R_2 is one. Thus

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$$

Count the number of operations

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Example

Let

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

What is $\mathbf{M}_{R_1 \cup R_2}$ and $\mathbf{M}_{R_1 \cap R_2}$

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Example

Let

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

What is $\mathbf{M}_{R_1 \cup R_2}$ and $\mathbf{M}_{R_1 \cap R_2}$

$$\mathbf{M}_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example

Let

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

What is $\mathbf{M}_{R_1 \cup R_2}$ and $\mathbf{M}_{R_1 \cap R_2}$

$$\mathbf{M}_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{M}_{R_1 \cap R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

How does combining the relations change their properties?

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

One can also compose relations easily with 0-1 matrices. If you have not seen matrix product before, you will need to read section 3.8.

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{R_1} \circ \mathbf{M}_{R_1} = \mathbf{M}_{R_1} \odot \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Latex notation: `\circ`, `\odot`.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Remember that recursively composing a relation $R^n, n = 1, 2, \dots$ gives a nice characterization of transitivity.

Using these ideas, we can build that Warshall (a.k.a. Roy-Warshall) algorithm for computing the *transitive closure* (discussed in the next section).

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

We will get more into graphs later on, but we briefly introduce them here since they can be used to represent relations.

In the general case, we have already seen directed graphs used to represent relations. However, for relations *on* a set A , it makes more sense to use a general graph rather than have *two* copies of the set in the diagram.

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Definition

A *graph* consists of a set V of *vertices* (or *nodes*) together with a set E of edges. We write $G = (V, E)$.

A *directed graph* (or *digraph*) consists of a set V of *vertices* (or *nodes*) together with a set E of edges of ordered pairs of elements of V .

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence
Relations

Again, a directed graph offers some insight as to the properties of a relation.

Reflexivity – In a digraph, a relation is reflexive if and only if every vertex has a self loop.

Symmetry – In a digraph, a represented relation is symmetric if and only if for every edge from x to y there is also a corresponding edge from y to x .

Relations

CSE235

Introduction

Properties

Representations

Matrices

Directed Graphs

Closures

Equivalence

Relations

Antisymmetry – A represented relation is antisymmetric if and only if there is never a back edge for each directed edge between distinct vertices.

Transitivity – A digraph is transitive if for every pair of edges (x, y) and (y, z) there is also a directed edge (x, z) (though this may be harder to verify in more complex graphs visually).

Relations

CSE235

Introduction

Properties

Representations

Closures

Definitions

Warshall's
AlgorithmEquivalence
Relations

If a given relation R is not reflexive (or symmetric, antisymmetric, transitive) can we transform it into a relation R' that is?

Example

Let $R = \{(1, 2), (2, 1), (2, 2), (3, 1), (3, 3)\}$ is not reflexive. How can we make it reflexive?

Closures

Definition

Relations

CSE235

Introduction

Properties

Representations

Closures

Definitions

Warshall's

Algorithm

Equivalence

Relations

If a given relation R is not reflexive (or symmetric, antisymmetric, transitive) can we transform it into a relation R' that is?

Example

Let $R = \{(1, 2), (2, 1), (2, 2), (3, 1), (3, 3)\}$ is not reflexive. How can we make it reflexive?

In general, we would like to change the relation as *little as possible*. To make this relation reflexive we simply have to add $(1, 1)$ to the set.

Relations

CSE235

Introduction

Properties

Representations

Closures

Definitions

Warshall's
AlgorithmEquivalence
Relations

If a given relation R is not reflexive (or symmetric, antisymmetric, transitive) can we transform it into a relation R' that is?

Example

Let $R = \{(1, 2), (2, 1), (2, 2), (3, 1), (3, 3)\}$ is not reflexive. How can we make it reflexive?

In general, we would like to change the relation as *little as possible*. To make this relation reflexive we simply have to add $(1, 1)$ to the set.

Inducing a property on a relation is called its *closure*. In the example, R' is the *reflexive closure*.

Relations

CSE235

Introduction

Properties

Representations

Closures

Definitions

Warshall's
AlgorithmEquivalence
Relations

In general, the reflexive closure of a relation R on A is $R \cup \Delta$ where

$$\Delta = \{(a, a) \mid a \in A\}$$

is the *diagonal relation* on A .

Question: How can we compute the reflexive closure using a 0-1 matrix representation? Digraph representation?

Similarly, we can create symmetric closures using the inverse of a relation. That is, $R \cup R^{-1}$ where

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

Question: How can we compute the symmetric closure using a 0-1 matrix representation? Digraph representation?

Also, transitive closures can be made using a previous theorem:

Theorem

A relation R is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \dots$

Thus, if we can compute R^k such that $R^k \subseteq R^n$ for all $n \geq k$, then R^k is the transitive closure.

To see how to efficiently do this, we present *Warshall's Algorithm*.

Note: your book gives much greater details in terms of graphs and *connectivity relations*. It is good to read these, but they are based on material that we have not yet seen.

Warshall's Algorithm I

Key Ideas

Relations

CSE235

Introduction

Properties

Representations

Closures

Definitions

Warshall's
AlgorithmEquivalence
Relations

In any set A with $|A| = n$ elements, any transitive relation will be built from a sequence of relations that has a length at most n . Why? Consider the case where A contains the relations

$$(a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, a_n)$$

Then (a_1, a_n) is required to be in A for A to be transitive.

Thus, by the previous theorem, it suffices to compute (*at most*) R^n . Recall that $R^k = R \circ R^{k-1}$ is calculated using a Boolean matrix product. This gives rise to a natural algorithm.

Relations

CSE235

Introduction

Properties

Representations

Closures

Definitions

Warshall's
AlgorithmEquivalence
Relations

WARSHALL'S ALGORITHM

INPUT : An $(n \times n)$ 0-1 Matrix \mathbf{M}_R representing a relation R
OUTPUT : A $(n \times n)$ 0-1 Matrix \mathbf{W} representing the transitive closure of R

```
1  $\mathbf{W} = \mathbf{M}_R$ 
2 FOR  $k = 1, \dots, n$  DO
3   FOR  $i = 1, \dots, n$  DO
4     FOR  $j = 1, \dots, n$  DO
5        $w_{i,j} = w_{i,j} \vee (w_{i,k} \wedge w_{k,j})$ 
6     END
7   END
8 END
9 return  $\mathbf{W}$ 
```

Relations

CSE235

Introduction

Properties

Representations

Closures

Definitions

Warshall's
AlgorithmEquivalence
Relations

Example

Compute the transitive closure of the relation

$$R = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 1), (3, 4), (4, 1), (4, 4)\}$$

on $A = \{1, 2, 3, 4\}$

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes
Examples

Consider the set of every person in the world. Now consider a relation such that $(a, b) \in R$ if a and b are siblings. Clearly, this relation is:

- reflexive,
- symmetric, and
- transitive.

Such a unique relation is called an *equivalence relation*.

Definition

A relation on a set A is an *equivalence relation* if it is reflexive, symmetric and transitive.

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Though a relation on a set A may not be an equivalence relation, we *can* define a subset of A such that R *does* become an equivalence relation (for that subset).

Definition

Let R be an equivalence relation on the set A and let $a \in A$. The set of all elements in A that are related to a is called the *equivalence class* of a . We denote this set $[a]_R$ (we omit R when there is no ambiguity as to the relation). That is,

$$[a]_R = \{s \mid (a, s) \in R, s \in A\}$$

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Elements in $[a]_R$ are called *representatives* of the equivalence class.

Theorem

Let R be an equivalence relation on a set A . The following are equivalent:

- 1 aRb
- 2 $[a] = [b]$
- 3 $[a] \cap [b] \neq \emptyset$

The proof in the book is a circular proof.

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Equivalence classes are important because they can *partition* a set A into disjoint non-empty subsets A_1, A_2, \dots, A_l where each equivalence class is self-contained.

Note that a partition satisfies these properties:

- $\bigcup_{i=1}^l A_i = A$
- $A_i \cap A_j = \emptyset$ for $i \neq j$
- $A_i \neq \emptyset$ for all i

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

For example, if R is a relation such that $(a, b) \in R$ if a and b live in the US and live in the same state, then R is an equivalence relation that *partitions* the set of people who live in the US into 50 equivalence classes.

Theorem

Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition A_i of the set S , there is an equivalence relation R that has the sets A_i as its equivalence classes.

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

In a 0-1 matrix, if the elements are ordered into their equivalence classes, equivalence classes/partitions form perfect squares of 1s (and zeros else where).

In a digraph, equivalence classes form a collection of disjoint *complete* graphs.

Example

Say that we have $A = \{1, 2, 3, 4, 5, 6, 7\}$ and R is an equivalence relation that partitions A into $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5, 6\}$ and $A_3 = \{7\}$. What does the 0-1 matrix look like? Digraph?

Equivalence Relations

Example I

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Let $R = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}$

- Reflexive?
- Transitive?
- Symmetric?

Equivalence Relations

Example I

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Let $R = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}$

- Reflexive?
- Transitive?
- Symmetric? No, it is not since, in particular $4 \leq 5$ but $5 \not\leq 4$.

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Let $R = \{(a, b) \mid a, b \in \mathbb{R}, a \leq b\}$

- Reflexive?
- Transitive?
- Symmetric? No, it is not since, in particular $4 \leq 5$ but $5 \not\leq 4$.
- Thus, R is not an equivalence relation.

Equivalence Relations

Example II

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Let $R = \{(a, b) \mid a, b \in \mathbb{Z}, a = b\}$

- Reflexive?
- Transitive?
- Symmetric?
- What are the equivalence classes that partition \mathbb{Z} ?

Equivalence Relations

Example III

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

For $(x, y), (u, v) \in \mathbb{R}^2$ define

$$R = \{((x, y), (u, v)) \mid x^2 + y^2 = u^2 + v^2\}$$

Show that R is an equivalence relation. What are the equivalence classes it defines (i.e. what are the partitions of \mathbb{R}^2 ?)

Equivalence Relations

Example IV

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Given $n, r \in \mathbb{N}$, define the set

$$n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}$$

Equivalence Relations

Example IV

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Given $n, r \in \mathbb{N}$, define the set

$$n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}$$

- For $n = 2, r = 0$, $2\mathbb{Z}$ represents the equivalence class of all even integers.

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Given $n, r \in \mathbb{N}$, define the set

$$n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}$$

- For $n = 2, r = 0$, $2\mathbb{Z}$ represents the equivalence class of all even integers.
- What n, r give the equivalence class of all *odd* integers?

Relations

CSE235

Introduction

Properties

Representations

Closures

Equivalence
RelationsEquivalence
Classes

Examples

Example

Given $n, r \in \mathbb{N}$, define the set

$$n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}$$

- For $n = 2, r = 0$, $2\mathbb{Z}$ represents the equivalence class of all even integers.
- What n, r give the equivalence class of all *odd* integers?
- If we set $n = 3, r = 0$ we get the equivalence class of all integers divisible by 3.

Example

Given $n, r \in \mathbb{N}$, define the set

$$n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}$$

- For $n = 2, r = 0$, $2\mathbb{Z}$ represents the equivalence class of all even integers.
- What n, r give the equivalence class of all *odd* integers?
- If we set $n = 3, r = 0$ we get the equivalence class of all integers divisible by 3.
- If we set $n = 3, r = 1$ we get the equivalence class of all integers divisible by 3 with a *remainder* of one.

Example

Given $n, r \in \mathbb{N}$, define the set

$$n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}$$

- For $n = 2, r = 0$, $2\mathbb{Z}$ represents the equivalence class of all even integers.
- What n, r give the equivalence class of all *odd* integers?
- If we set $n = 3, r = 0$ we get the equivalence class of all integers divisible by 3.
- If we set $n = 3, r = 1$ we get the equivalence class of all integers divisible by 3 with a *remainder* of one.
- In general, this relation defines equivalence classes that are, in fact, *congruence classes*. (see Section 3.4, to be covered later).