Predicate Logic and Quantifiers

Slides by Christopher M. Bourke
Instructor: Berthe Y. Choueiry

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Computer Science & Engineering 235
Introduction to Discrete Mathematics
Sections 1.3–1.4 of Rosen

cse235@cse.unl.edu
Consider the following statements:

\[ x > 3, \quad x = y + 3, \quad x + y = z \]

The truth value of these statements has no meaning without specifying the values of \( x, y, z \).

However, we *can* make propositions out of such statements.

A **predicate** is a property that is affirmed or denied about the **subject** (in logic, we say “variable” or “argument”) of a **statement**.

\[ \text{“} \underbrace{x}_{\text{subject}} \overset{\text{is greater than 3}}{\text{is greater than 3”}} \underbrace{3}_{\text{predicate}} \]
Propositional Functions

To write in predicate logic:

“\( x \) is greater than 3”

We introduce a (functional) symbol for the predicate, and put
the subject as an argument (to the functional symbol): \( P(x) \)

Examples:

- Father(\( x \)): unary predicate
- Brother(\( x,y \)): binary predicate
- Sum(\( x,y,z \)): ternary predicate
- P(\( x,y,z,t \)): \( n \)-ary predicate
Propositional Functions

Definition

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function $P$. Here, $(x_1, x_2, \ldots, x_n)$ is an $n$-tuple and $P$ is a predicate.

You can think of a propositional function as a function that

- Evaluates to true or false.
- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.
Example

Let $Q(x, y, z)$ denote the statement \(x^2 + y^2 = z^2\). What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of $(x, y, z)$ make the predicate true?

Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.

Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for $(x, y, z)$ that make this propositional function true—how many right triangles are there?
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Consider the previous example. Does it make sense to assign to $x$ the value “blue”?

Intuitively, the universe of discourse is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function $P(x) = “The test will be on $x$ the 23rd” be?
Moreover, each variable in an \( n \)-tuple may have a different universe of discourse.

Let \( P(r, g, b, c) = \text{“The rgb-value of the color } c \text{ is } (r, g, b)” \).

For example, \( P(255, 0, 0, \text{red}) \) is true, while \( P(0, 0, 255, \text{green}) \) is false.

What are the universes of discourse for \((r, g, b, c)\)?
A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to quantify it. That is, the predicate is true (or false) for all possible values in the universe of discourse or for some value(s) in the universe of discourse.

Such quantification can be done with two quantifiers: the universal quantifier and the existential quantifier.
Universal Quantifier

Definition

The *universal quantification* of a predicate $P(x)$ is the proposition “$P(x)$ is true for all values of $x$ in the universe of discourse” We use the notation

$$\forall x P(x)$$

which can be read “for all $x$”

If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of all elements:

$$\forall x P(x) \iff P(n_1) \land P(n_2) \land \cdots \land P(n_k)$$
Universal Quantifier
Example 1

- Let $P(x)$ be the predicate “$x$ must take a discrete mathematics course” and let $Q(x)$ be the predicate “$x$ is a computer science student”.
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statement “Every computer science student must take a discrete mathematics course”.

Express the statement “Everybody must take a discrete mathematics course or be a computer science student”.

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  \[ \forall x (Q(x) \lor P(x)) \]
- Are these statements true or false?
Express the statement “for every $x$ and for every $y$, $x + y > 10$”
Universal Quantifier
Example II

Express the statement “for every $x$ and for every $y$, $x + y > 10$”

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for $x, y$ is the set of integers.
Universal Quantifier

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Answer:

$$\forall x \forall y P(x, y)$$
Universal Quantifier

Example II

Express the statement “for every \( x \) and for every \( y \), \( x + y > 10 \)”

Let \( P(x, y) \) be the statement \( x + y > 10 \) where the universe of discourse for \( x, y \) is the set of integers.

Answer:

\[ \forall x \forall y P(x, y) \]

Note that we can also use the shorthand

\[ \forall x, y P(x, y) \]
Existential Quantifier

Definition

The existential quantification of a predicate $P(x)$ is the proposition “There exists an $x$ in the universe of discourse such that $P(x)$ is true.” We use the notation

$$\exists x P(x)$$

which can be read “there exists an $x$”

Again, if the universe of discourse is finite, $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of all elements:

$$\exists x P(x) \iff P(n_1) \lor P(n_2) \lor \cdots \lor P(n_k)$$
Existential Quantifier
Example 1

Let \( P(x, y) \) denote the statement, “\( x + y = 5 \)”.
What does the expression,

\[
\exists x \exists y P(x, y)
\]

mean?
What universe(s) of discourse make it true?
Express the statement “there exists a real solution to $ax^2 + bx - c = 0$"
Existential Quantifier
Example II

Express the statement “there exists a real solution to \( ax^2 + bx - c = 0 \)”

Let \( P(x) \) be the statement \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) where the universe of discourse for \( x \) is the set of reals. Note here that \( a, b, c \) are all fixed constants.
Express the statement “there exists a real solution to \( ax^2 + bx - c = 0 \)”

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The statement can thus be expressed as

\[ \exists x P(x) \]
Question: what is the truth value of $\exists x P(x)$?
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Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such real number $x$ can satisfy the predicate.

How can we make it so that it is true?
Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such real number $x$ can satisfy the predicate.

How can we make it so that it is true?

Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$. 
In general, when are quantified statements true/false?

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**Table:** Truth Values of Quantifiers
Existential and universal quantifiers can be used together to quantify a predicate statement; for example,

$$\forall x \exists y P(x, y)$$

is perfectly valid. However, you must be careful—it must be read left to right.

For example, $$\forall x \exists y P(x, y)$$ is not equivalent to $$\exists y \forall x P(x, y)$$. Thus, ordering is important.
Mixing Quantifiers II

For example:

- $\forall x \exists y \text{Loves}(x, y)$: everybody loves somebody
- $\exists y \forall x \text{Loves}(x, y)$: There is someone loved by everyone

Those expressions do not mean the same thing!

Note that $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$, but the converse does not hold

However, you can commute similar quantifiers; $\exists x \exists y P(x, y)$ is equivalent to $\exists y \exists x P(x, y)$ (which is why our shorthand was valid).
# Mixing Quantifiers

## Truth Values

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**Table:** Truth Values of 2-variate Quantifiers
Mixing Quantifiers

Example 1

Express, in predicate logic, the statement that there are an infinite number of integers.
Mixing Quantifiers

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Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$. 
Mixing Quantifiers

Example I

Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following.

$$\forall x \exists y P(x, y)$$
Express the *commutative law of addition* for $\mathbb{R}$.
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We want to express that for every pair of reals, $x, y$ the following identity holds:

$$x + y = y + x$$
Express the *commutative law of addition* for $\mathbb{R}$.

We want to express that for every pair of reals, $x, y$ the following identity holds:

$$x + y = y + x$$

Then we have the following:

$$\forall x \forall y (x + y = y + x)$$
Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.
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We want to express that for every real number \( x \), there exists a real number \( y \) such that \( xy = 1 \).
Mixing Quantifiers
Example II: More Mathematical Statements Continued

Express the *multiplicative inverse law* for (nonzero) rationals \( \mathbb{Q} \setminus \{0\} \).

We want to express that for every real number \( x \), there exists a real number \( y \) such that \( xy = 1 \).

Then we have the following:

\[ \forall x \exists y (xy = 1) \]
Is commutativity for subtraction valid over the reals?
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That is, for all pairs of real numbers \( x, y \) does the identity \( x - y = y - x \) hold? Express this using quantifiers.
Is commutativity for subtraction valid over the reals?
That is, for all pairs of real numbers $x, y$ does the identity $x - y = y - x$ hold? Express this using quantifiers.

The expression is

$$\forall x \forall y (x - y = y - x)$$
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Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer $x$ does there exists a $y$ such that $xy = 1$?

This is false, since we can find a *counter example*. Take any integer, say 5 and multiply it with another integer, $y$. If the statement held, then $5 = 1/y$, but for any (nonzero) integer $y$, $|1/y| \leq 1$. 
Express the statement “there is a number $x$ such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is $x$” as a logical expression.

Solution:
Express the statement “there is a number $x$ such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is $x$” as a logical expression.

Solution:

- Let $P(x, y)$ be the expression “$x + y = y$".
Express the statement "there is a number \( x \) such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is \( x \)" as a logical expression.

Solution:

- Let \( P(x, y) \) be the expression "\( x + y = y \)".
- Let \( Q(x, y) \) be the expression "\( xy = x \)".
Express the statement “there is a number \( x \) such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is \( x \)” as a logical expression.

Solution:

- Let \( P(x, y) \) be the expression “\( x + y = y \)”.
- Let \( Q(x, y) \) be the expression “\( xy = x \)”.
- Then the expression is

\[
\exists x \forall y \left( P(x, y) \land Q(x, y) \right)
\]
Express the statement “there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x” as a logical expression.

Solution:

- Let $P(x, y)$ be the expression “$x + y = y$”.
- Let $Q(x, y)$ be the expression “$xy = x$”.
- Then the expression is

$$\exists x \forall y (P(x, y) \land Q(x, y))$$

- Over what universe(s) of discourse does this statement hold?
Express the statement “there is a number \( x \) such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is \( x \)” as a logical expression.

Solution:

- Let \( P(x, y) \) be the expression “\( x + y = y \)”.
- Let \( Q(x, y) \) be the expression “\( xy = x \)”.
- Then the expression is

\[ \exists x \forall y (P(x, y) \land Q(x, y)) \]

- Over what universe(s) of discourse does this statement hold?
- This is the additive identity law and holds for \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q} \) but does not hold for \( \mathbb{Z}^+ \).
When a quantifier is used on a variable $x$, we say that $x$ is \textit{bound}. If no quantifier is used on a variable in a predicate statement, it is called \textit{free}.

\textbf{Example}

In the expression $\exists x \forall y P(x, y)$ both $x$ and $y$ are bound.
In the expression $\forall x P(x, y)$, $x$ is bound, but $y$ is free.

A statement is called a \textit{well-formed formula}, when all variables are properly quantified.
The set of all variables bound by a common quantifier is the \textit{scope} of that quantifier.

\begin{example}
In the expression $\exists x, y \forall z P(x, y, z, c)$ the scope of the existential quantifier is $\{x, y\}$, the scope of the universal quantifier is just $z$ and $c$ has no scope since it is free.
\end{example}
Negation

Just as we can use negation with propositions, we can use them with quantified expressions.

**Lemma**

Let $P(x)$ be a predicate. Then the following hold.

\[
\neg \forall x P(x) \equiv \exists x \neg P(x)
\]

\[
\neg \exists x P(x) \equiv \forall x \neg P(x)
\]

This is essentially a quantified version of De Morgan’s Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan’s law).
## Negation

### Truth Values

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**Table:** Truth Values of Negated Quantifiers
Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Calculus. It was developed by the logicians of the artificial intelligence community for symbolic reasoning.

- Prolog allows the user to express facts and rules.
- Facts are propositional functions: student(juana), enrolled(juana,cse235), instructor(patel,cse235), etc.
- Rules are implications with conjunctions:
  teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as:
  ?enrolled(juana,cse478)
  ?enrolled(X,cse478)
  ?teaches(X,juana)
by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5.
Logic is more precise than English.

Transcribing English to Logic and vice versa can be tricky.

When writing statements with quantifiers, *usually* the correct meaning is conveyed with the following combinations:

- **Use $\forall$ with $\Rightarrow$**
  - Example: $\forall x \text{Lion}(x) \Rightarrow \text{Fierce}(x)$
  - $\forall x \text{Lion}(x) \land \text{Fierce}(x)$ means “everyone is a lion and everyone is fierce”

- **Use $\exists$ with $\land$**
  - Example: $\exists x \text{Lion}(x) \land \text{Drinks}(x, \text{coffee})$: holds when you have at least one lion that drinks coffee
  - $\exists x \text{Lion}(x) \Rightarrow \text{Drinks}(x, \text{coffee})$ holds when you have people even though no lion drinks coffee.
Examples? Exercises?

- Rewrite the expression,
  \[ \neg \forall x (\exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z)) \]

- Let \( P(x, y) \) denote “\( x \) is a factor of \( y \)” where \( x \in \{1, 2, 3, \ldots \} \) and \( y \in \{2, 3, 4, \ldots \} \). Let \( Q(y) \) denote “\( \forall x [P(x, y) \rightarrow ((x = y) \lor (x = 1))] \)”.
  When is \( Q(y) \) true?
Examples? Exercises?

- Rewrite the expression,
  \[ \neg \forall x (\exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z)) \]

- Answer: Use the negated quantifiers and De Morgan’s law.
  \[ \exists x (\forall y \exists z \neg P(x, y, z) \lor \forall z \exists y \neg P(x, y, z)) \]

- Let \( P(x, y) \) denote “\( x \) is a factor of \( y \)” where \( x \in \{1, 2, 3, \ldots\} \) and \( y \in \{2, 3, 4, \ldots\} \). Let \( Q(y) \) denote
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- Rewrite the expression,
  \[ \neg \forall x (\exists y \forall z P(x, y, z) \land \exists z \forall y P(x, y, z)) \]

  Answer: Use the negated quantifiers and De Morgan’s law.

  \[ \exists x (\forall y \exists z \neg P(x, y, z) \lor \forall z \exists y \neg P(x, y, z)) \]

- Let \( P(x, y) \) denote “\( x \) is a factor of \( y \)” where \( x \in \{1, 2, 3, \ldots\} \) and \( y \in \{2, 3, 4, \ldots\} \). Let \( Q(y) \) denote “\( \forall x [P(x, y) \rightarrow ((x = y) \lor (x = 1))] \)” . When is \( Q(y) \) true?

  Answer: Only when \( y \) is a prime number.
Some students wondered if

$$\forall x, y P(x, y) \equiv \forall x P(x, y) \land \forall y P(x, y)$$

This is certainly not true. In the left-hand side, both $x$ and $y$ are bound. In the right-hand side, $x$ is bound in the first predicate, but $y$ is free. In the second predicate, $y$ is bound but $x$ is free.

All variables that occur in a propositional function must be bound to turn it into a proposition.

Thus, the left-hand side is a proposition, but the right-hand side is not. How can they be equivalent?