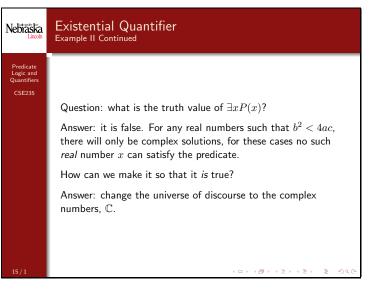


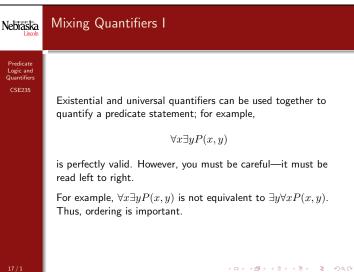
Existential Quantifier Nebraska Question: what is the truth value of $\exists x P(x)$? Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such real number x can satisfy the predicate. How can we make it so that it is true?

Quantifiers

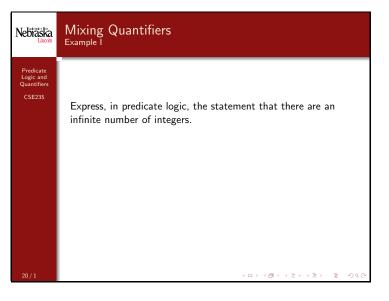
Nebraska Truth Values In general, when are quantified statements true/false? False When Statement True When There is an x for $\forall x P(x)$ P(x) is true for every which P(x) is false. $\exists x P(x)$ $\overline{\text{There is an}} \quad x \quad \text{for} \quad$ P(x) is false for every which P(x) is true. Table: Truth Values of Quantifiers

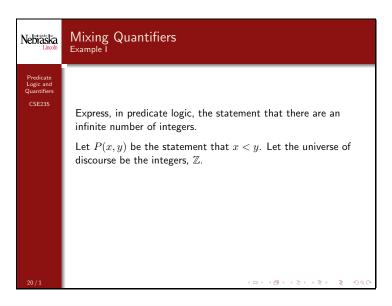
Nebraska Lincoln	Mixing Quantifiers II
Predicate Logic and Quantifiers CSE235	For example:
	Those expressions do not mean the same thing! Note that $\exists y \forall x P(x,y) \to \forall x \exists y P(x,y)$, but the converse does not hold However, you <i>can</i> commute <i>similar</i> quantifiers; $\exists x \exists y P(x,y)$ is equivalent to $\exists y \exists x P(x,y)$ (which is why our shorthand was
18/1	valid).

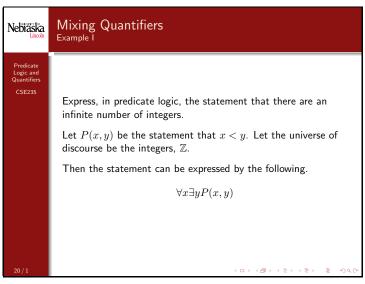


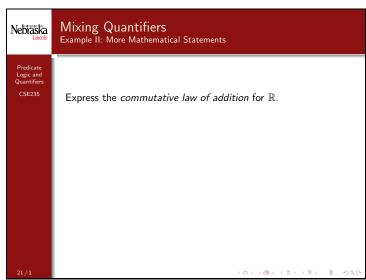


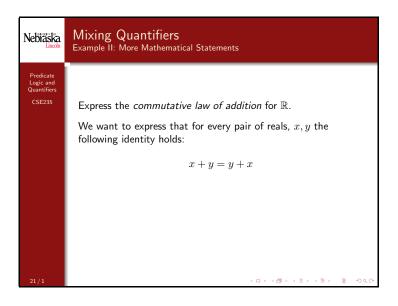
Nebraska Lincoln	Mixing Quantifiers Truth Values				
Predicate	Statement	True When	False When		
Logic and Quantifiers	$\forall x \forall y P(x,y)$	P(x,y) is true for ev-	There is at least one		
CSE235		ery pair x, y .	pair, x,y for which		
			P(x,y) is false.		
	$\forall x \exists y P(x,y)$	For every x , there is a	There is an x for		
		y for which $P(x,y)$ is	which $P(x,y)$ is false		
		true.	for every y .		
	$\exists x \forall y P(x,y)$	There is an x for	For every x , there is a		
		which $P(x,y)$ is true	y for which $P(x,y)$ is		
		for every y .	false.		
	$\exists x \exists y P(x,y)$	There is at least one	P(x,y) is false for ev-		
		pair x,y for which	ery pair x, y .		
		P(x,y) is true.			
	Table: Truth Values of 2-variate Quantifiers				
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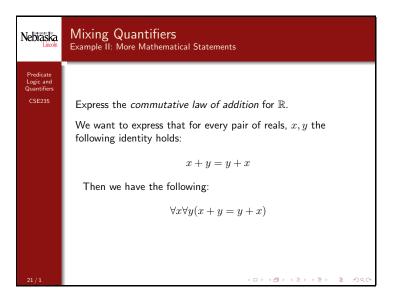


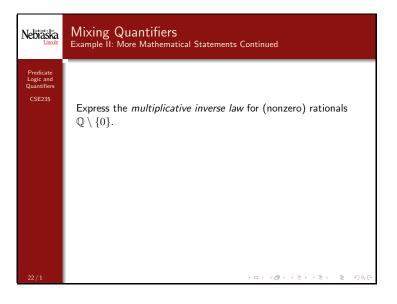


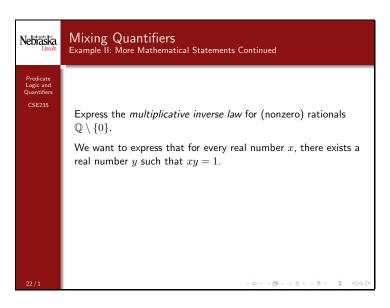


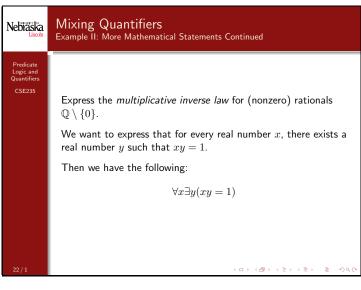


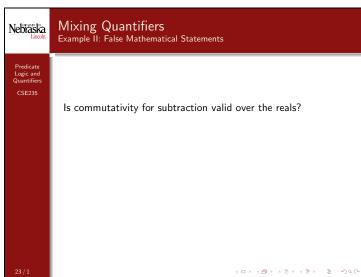


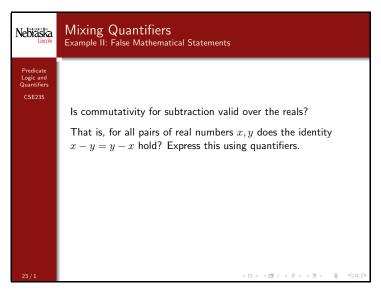


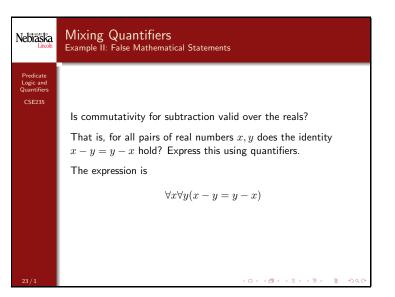


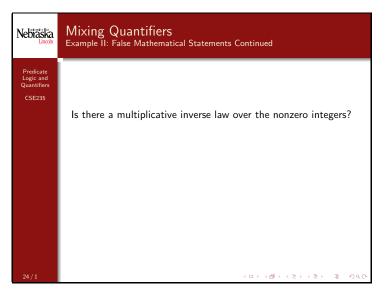


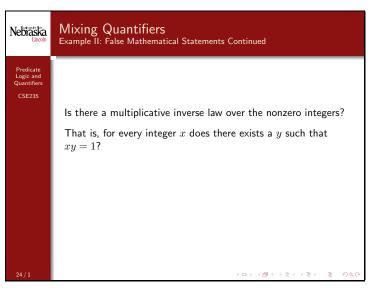


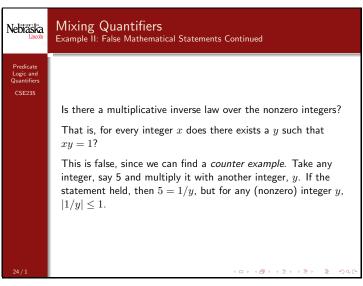


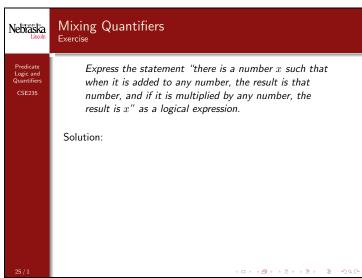


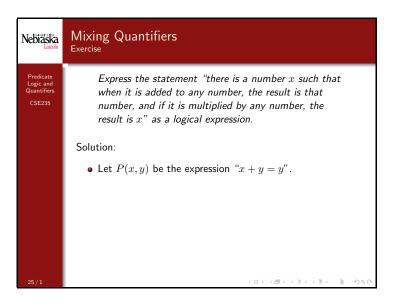


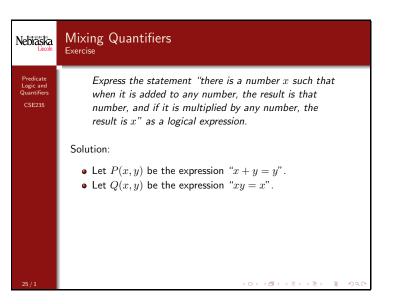














Mixing Quantifiers

Predicate Logic and

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let P(x,y) be the expression "x + y = y".
- Let Q(x,y) be the expression "xy = x".
- Then the expression is

$$\exists x \forall y (P(x,y) \land Q(x,y))$$

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Mixing Quantifiers

Predicate Logic and Quantifiers CSE235 Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

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 Over what universe(s) of discourse does this statement hold?

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Mixing Quantifiers

Predicate Logic and Quantifiers CSE235 Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let P(x,y) be the expression "x+y=y".
- Let Q(x,y) be the expression "xy = x".
- Then the expression is

$$\exists x \forall y (P(x,y) \land Q(x,y))$$

- Over what universe(s) of discourse does this statement hold?
- This is the additive identity law and holds for $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$ but does not hold for \mathbb{Z}^+ .





Binding Variables I

Predicate Logic and Quantifier CSE235

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When a quantifier is used on a variable x, we say that x is bound. If no quantifier is used on a variable in a predicate statement, it is called *free*.

Example

In the expression $\exists x \forall y P(x,y)$ both x and y are bound. In the expression $\forall x P(x,y), \ x$ is bound, but y is free.

A statement is called a *well-formed formula*, when all variables are properly quantified.

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Binding Variables II

Predicate Logic and Quantifiers

The set of all variables bound by a common quantifier is the *scope* of that quantifier.

Example

In the expression $\exists x,y \forall z P(x,y,z,c)$ the scope of the existential quantifier is $\{x,y\}$, the scope of the universal quantifier is just z and c has no scope since it is free.





Negation

Logic and Quantifiers

CSE235

Just as we can use negation with propositions, we can use them with quantified expressions.

Lemma

Let P(x) be a predicate. Then the following hold.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).

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Negation

Predicate Logic and Quantifiers CSE235

Statement	True When	False When
$\neg \exists x P(x) \equiv$	For every x , $P(x)$ is	There is an x for
$\forall x \neg P(x)$	false.	which $P(x)$ is true.
$\neg \forall x P(x) \equiv$	There is an x for	P(x) is true for every
$\exists x \neg P(x)$	which $P(x)$ is false.	x.

Table: Truth Values of Negated Quantifiers



English into Logic

Predicate Logic and Quantifiers CSE235 Logic is more precise than English.

Transcribing English to Logic and vice versa can be tricky.

When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:

- Use \forall with \Rightarrow Example: $\forall x Lion(x) \Rightarrow Fierce(x)$
 - $\forall xLion(x) \land Fierce(x)$ means "everyone is a lion and everyone is fierce"
- everyone is herce
- $\bullet \ \ \mathsf{Use} \ \exists \ \mathsf{with} \ \land \\$

Example: $\exists x Lion(x) \land Drinks(x, coffee)$: holds when you have at least one lion that drinks coffee $\exists x Lion(x) \Rightarrow Drinks(x, coffee)$ holds when you have people even though no lion drinks coffee.

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Prolog

Predicate Logic and Quantifiers CSE235 Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Calculus. It was developed by the logicians of the artificial intelligence community for symbolic reasoning.

- Prolog allows the user to express facts and rules
- Facts are proposational functions: student(juana), enrolled(juana,cse235), instructor(patel,cse235), etc.
- Rules are implications with conjunctions: teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as: ?enrolled(juana,cse478)
 ?enrolled(X,cse478)

?teaches(X,juana)

by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5.

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Conclusion

Logic and Quantifiers CSE235

Examples? Exercises?

- $\begin{array}{l} \bullet \ \ \text{Rewrite the expression,} \\ \neg \forall x \big(\exists y \forall z P(x,y,z) \land \exists z \forall y P(x,y,z) \big) \end{array}$
- \bullet Let P(x,y) denote "x is a factor of y " where $x\in\{1,2,3,\ldots\}$ and $y\in\{2,3,4,\ldots\}.$ Let Q(y) denote " $\forall x\big[P(x,y)\to((x=y)\vee(x=1))\big]$ ". When is Q(y) true?

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Conclusion



Examples? Exercises?

- Rewrite the expression, $\neg \forall x \big(\exists y \forall z P(x,y,z) \land \exists z \forall y P(x,y,z) \big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x \big(\forall y \exists z \neg P(x, y, z) \lor \forall z \exists y \neg P(x, y, z) \big)$$

 \bullet Let P(x,y) denote "x is a factor of y " where $x\in\{1,2,3,\ldots\}$ and $y\in\{2,3,4,\ldots\}.$ Let Q(y) denote " $\forall x\big[P(x,y)\to((x=y)\vee(x=1))\big]$ ". When is Q(y) true?

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Conclusion

Logic and Quantifier CSE235

Examples? Exercises?

- Rewrite the expression, $\neg \forall x \big(\exists y \forall z P(x,y,z) \land \exists z \forall y P(x,y,z)\big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x (\forall y \exists z \neg P(x, y, z) \lor \forall z \exists y \neg P(x, y, z))$$

- Let P(x,y) denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let Q(y) denote " $\forall x \big[P(x,y) \to ((x=y) \lor (x=1)) \big]$ ". When is Q(y) true?
- \bullet Answer: Only when y is a prime number.

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Extra Question

Predicate Logic and Quantifiers CSE235

Some students wondered if

$$\forall x, y P(x, y) \equiv \forall x P(x, y) \land \forall y P(x, y)$$

This is certainly not true. In the left-hand side, both x and y are bound. In the right-hand side, x is bound in the first predicate, but y is free. In the second predicate, y is bound but x is free.

All variables that occur in a propositional function must be bound to turn it into a proposition.

Thus, the left-hand side is a proposition, but the right-hand side is not. How can they be equivalent?

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