Predicate Logic and Quantifiers

CSE235

Predicate Logic and Quantifiers

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Fall 2007

Computer Science & Engineering 235
Introduction to Discrete Mathematics
Sections 1.3–1.4 of Rosen

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Notes

Introduction

Consider the following statements:

$$x > 3, \quad x = y + 3, \quad x + y = z$$

The truth value of these statements has no meaning without specifying the values of x, y, z.

However, we *can* make propositions out of such statements.

A *predicate* is a property that is affirmed or denied about the *subject* (in logic, we say "variable" or "argument") of a *statement*.

$$\text{“}\underbrace{x}_{\text{subject}}\ \underbrace{\text{is greater than 3”}}_{\text{predicate}}$$

Terminology: affirmed = holds = is true; denied = does not hold = is not true.

Notes

Propositional Functions

To write in predicate logic:

$$\text{“}\underbrace{x}_{\text{subject}}\ \underbrace{\text{is greater than 3”}}_{\text{predicate}}$$

We introduce a (functional) symbol for the predicate, and put the subject as an argument (to the functional symbol): $P(x)$

Examples:

- Father(x): unary predicate
- Brother(x,y): binary predicate
- Sum(x,y,z): ternary predicate
- P(x,y,z,t): n-ary predicate

Notes

Propositional Functions

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Definition

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the *propositional function* P. Here, $(x_1, x_2, \ldots, x_n)$ is an n-tuple and P is a predicate.

You can think of a propositional function as a function that

- Evaluates to true or false.
- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.

Notes

Propositional Functions
Example

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Example

Let $Q(x, y, z)$ denote the statement "$x^2 + y^2 = z^2$". What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of (x, y, z) make the predicate true?

Notes

Propositional Functions
Example

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Example

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Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.

Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for (x, y, z) that make this propositional function true—how many right triangles are there?

Notes

Universe of Discourse

Consider the previous example. Does it make sense to assign to x the value "blue"?

Intuitively, the *universe of discourse* is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function $P(x) =$ "The test will be on x the 23rd" be?

Notes

Universe of Discourse

Multivariate Functions

Moreover, each variable in an n-tuple may have a different universe of discourse.

Let $P(r, g, b, c) =$ "The rgb-value of the color c is (r, g, b)".

For example, $P(255, 0, 0, red)$ is true, while $P(0, 0, 255, green)$ is false.

What are the universes of discourse for (r, g, b, c)?

Notes

Quantifiers

Introduction

A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to *quantify* it. That is, the predicate is true (or false) for *all* possible values in the universe of discourse or for *some* value(s) in the universe of discourse.

Such *quantification* can be done with two *quantifiers*: the *universal* quantifier and the *existential* quantifier.

Notes

Universal Quantifier
Definition

Predicate Logic and Quantifiers

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Definition

The *universal quantification* of a predicate $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of all elements:

$$\forall x P(x) \iff P(n_1) \wedge P(n_2) \wedge \cdots \wedge P(n_k)$$

Notes

Universal Quantifier
Example I

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- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".
- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

Notes

Universal Quantifier
Example I

Predicate Logic and Quantifiers

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- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".

$$\forall x (Q(x) \rightarrow P(x))$$

- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

Notes

Universal Quantifier

Example I

Predicate Logic and Quantifiers

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$$\forall x(Q(x) \vee P(x))$$

Notes

Universal Quantifier

Example I

Predicate Logic and Quantifiers

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$$\forall x(Q(x) \vee P(x))$$

- Are hetse statements true or false?

Notes

Universal Quantifier

Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

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Universal Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

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Universal Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Notes

Universal Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Note that we can also use the shorthand

$$\forall x, y P(x, y)$$

Notes

Existential Quantifier
Definition

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Again, if the universe of discourse is finite, $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of all elements:

$$\exists x P(x) \iff P(n_1) \vee P(n_2) \vee \cdots \vee P(n_k)$$

Notes

Existential Quantifier
Example I

Let $P(x, y)$ denote the statement, "$x + y = 5$".

What does the expression,

$$\exists x \exists y P(x, y)$$

mean?

What universe(s) of discourse make it true?

Notes

Existential Quantifier
Example II

Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Notes

Existential Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

Notes

Existential Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

The statement can thus be expressed as

$$\exists x P(x)$$

Notes

Existential Quantifier
Example II Continued

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Question: what is the truth value of $\exists x P(x)$?

Notes

Existential Quantifier
Example II Continued

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Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such *real* number x can satisfy the predicate.

How can we make it so that it *is* true?

Notes

Existential Quantifier
Example II Continued

Predicate Logic and Quantifiers

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Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such *real* number x can satisfy the predicate.

How can we make it so that it *is* true?

Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$.

Notes

Quantifiers
Truth Values

Predicate Logic and Quantifiers

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In general, when are quantified statements true/false?

Statement	True When	False When
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Table: Truth Values of Quantifiers

Notes

Mixing Quantifiers I

Existential and universal quantifiers can be used together to quantify a predicate statement; for example,

$$\forall x \exists y P(x, y)$$

is perfectly valid. However, you must be careful—it must be read left to right.

For example, $\forall x \exists y P(x, y)$ is not equivalent to $\exists y \forall x P(x, y)$. Thus, ordering is important.

Notes

Mixing Quantifiers II

For example:

- $\forall x \exists y Loves(x, y)$: everybody loves somebody
- $\exists y \forall x Loves(x, y)$: There is someone loved by everyone

Those expressions do not mean the same thing!

Note that $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$, but the converse does not hold

However, you *can* commute *similar* quantifiers; $\exists x \exists y P(x, y)$ is equivalent to $\exists y \exists x P(x, y)$ (which is why our shorthand was valid).

Notes

Mixing Quantifiers
Truth Values

Statement	True When	False When
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair, x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table: Truth Values of 2-variate Quantifiers

Notes

Mixing Quantifiers
Example I

Express, in predicate logic, the statement that there are an infinite number of integers.

Notes

Mixing Quantifiers
Example I

Predicate Logic and Quantifiers
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Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Notes

Mixing Quantifiers
Example I

Predicate Logic and Quantifiers
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Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following.

$$\forall x \exists y P(x, y)$$

Notes

Mixing Quantifiers

Example II: More Mathematical Statements

Express the *commutative law of addition* for $\mathbb{R}$.

Notes

Mixing Quantifiers

Example II: More Mathematical Statements

Express the *commutative law of addition* for $\mathbb{R}$.

We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Notes

Mixing Quantifiers

Example II: More Mathematical Statements

Express the *commutative law of addition* for $\mathbb{R}$.

We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Then we have the following:

$$\forall x \forall y (x + y = y + x)$$

Notes

Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

Notes

Mixing Quantifiers
Example II: More Mathematical Statements Continued

Predicate Logic and Quantifiers
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Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

We want to express that for every real number x, there exists a real number y such that $xy = 1$.

Notes

Mixing Quantifiers
Example II: More Mathematical Statements Continued

Predicate Logic and Quantifiers
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Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

We want to express that for every real number x, there exists a real number y such that $xy = 1$.

Then we have the following:

$$\forall x \exists y (xy = 1)$$

Notes

Mixing Quantifiers
Example II: False Mathematical Statements

Predicate Logic and Quantifiers

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Is commutativity for subtraction valid over the reals?

Notes

Mixing Quantifiers
Example II: False Mathematical Statements

Predicate Logic and Quantifiers

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Is commutativity for subtraction valid over the reals?

That is, for all pairs of real numbers x, y does the identity $x - y = y - x$ hold? Express this using quantifiers.

Notes

Mixing Quantifiers
Example II: False Mathematical Statements

Predicate Logic and Quantifiers

CSE235

Is commutativity for subtraction valid over the reals?

That is, for all pairs of real numbers x, y does the identity $x - y = y - x$ hold? Express this using quantifiers.

The expression is

$$\forall x \forall y (x - y = y - x)$$

Notes

Mixing Quantifiers
Example II: False Mathematical Statements Continued

Is there a multiplicative inverse law over the nonzero integers?

Notes

Nebraska Lincoln

Mixing Quantifiers
Example II: False Mathematical Statements Continued

Predicate Logic and Quantifiers
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Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer x does there exists a y such that $xy = 1$?

Notes

Nebraska Lincoln

Mixing Quantifiers
Example II: False Mathematical Statements Continued

Predicate Logic and Quantifiers
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Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer x does there exists a y such that $xy = 1$?

This is false, since we can find a *counter example*. Take any integer, say 5 and multiply it with another integer, y. If the statement held, then $5 = 1/y$, but for any (nonzero) integer y, $|1/y| \leq 1$.

Notes

Mixing Quantifiers
Exercise

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers
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Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers
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Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

CSE235

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".
- Then the expression is

$$\exists x \forall y \, (P(x, y) \wedge Q(x, y))$$

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

CSE235

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".
- Then the expression is

$$\exists x \forall y \, (P(x, y) \wedge Q(x, y))$$

- Over what universe(s) of discourse does this statement hold?

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

CSE235

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".
- Then the expression is

$$\exists x \forall y \, (P(x, y) \wedge Q(x, y))$$

- Over what universe(s) of discourse does this statement hold?

- This is the *additive identity law* and holds for $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$ but does not hold for $\mathbb{Z}^+$.

Notes

Binding Variables I

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When a quantifier is used on a variable x, we say that x is *bound*. If no quantifier is used on a variable in a predicate statement, it is called *free*.

Example

In the expression $\exists x \forall y P(x,y)$ both x and y are bound.
In the expression $\forall x P(x,y)$, x is bound, but y is free.

A statement is called a *well-formed formula*, when all variables are properly quantified.

Notes

Binding Variables II

Predicate Logic and Quantifiers

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The set of all variables bound by a common quantifier is the *scope* of that quantifier.

Example

In the expression $\exists x, y \forall z P(x,y,z,c)$ the scope of the existential quantifier is $\{x,y\}$, the scope of the universal quantifier is just z and c has no scope since it is free.

Notes

Negation

Predicate Logic and Quantifiers

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Just as we can use negation with propositions, we can use them with quantified expressions.

Lemma

Let $P(x)$ be a predicate. Then the following hold.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).

Notes

Negation
Truth Values

Predicate Logic and Quantifiers

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Statement	True When	False When
$\neg\exists x P(x) \equiv \forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x) \equiv \exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Table: Truth Values of Negated Quantifiers

Notes

Prolog

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Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Calculus. It was developped by the logicians of the artificial intelligence community for symbolic reasoning.

- Prolog allows the user to express facts and rules
- Facts are proposational functions: student(juana), enrolled(juana,cse235), instructor(patel,cse235), etc.
- Rules are implications with conjunctions: teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as:
  ?enrolled(juana,cse478)
  ?enrolled(X,cse478)
  ?teaches(X,juana)
  by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5.

Notes

English into Logic

Predicate Logic and Quantifiers

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Logic is more precise than English.

Transcribing English to Logic and vice versa can be tricky.

When writing statements with quantifiers, *usually* the correct meaning is conveyed with the following combinations:

- Use $\forall$ with $\Rightarrow$
  Example: $\forall x Lion(x) \Rightarrow Fierce(x)$
  $\forall x Lion(x) \wedge Fierce(x)$ means "everyone is a lion and everyone is fierce"
- Use $\exists$ with $\wedge$
  Example: $\exists x Lion(x) \wedge Drinks(x, coffee)$: holds when you have at least one lion that drinks coffee
  $\exists x Lion(x) \Rightarrow Drinks(x, coffee)$ holds when you have people even though no lion drinks coffee.

Notes

Conclusion

Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y\forall z P(x,y,z) \wedge \exists z\forall y P(x,y,z)\big)$
- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \rightarrow ((x=y) \vee (x=1))\big]$". When is $Q(y)$ true?

Notes

Conclusion

Predicate Logic and Quantifiers
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Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y\forall z P(x,y,z) \wedge \exists z\forall y P(x,y,z)\big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x\big(\forall y\exists z\neg P(x,y,z) \vee \forall z\exists y\neg P(x,y,z)\big)$$

- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \rightarrow ((x=y) \vee (x=1))\big]$". When is $Q(y)$ true?

Notes

Conclusion

Predicate Logic and Quantifiers
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Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y\forall z P(x,y,z) \wedge \exists z\forall y P(x,y,z)\big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x\big(\forall y\exists z\neg P(x,y,z) \vee \forall z\exists y\neg P(x,y,z)\big)$$

- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \rightarrow ((x=y) \vee (x=1))\big]$". When is $Q(y)$ true?
- Answer: Only when y is a prime number.

Notes

Extra Question

Some students wondered if

$$\forall x, y P(x, y) \equiv \forall x P(x, y) \wedge \forall y P(x, y)$$

This is certainly not true. In the left-hand side, both x and y are bound. In the right-hand side, x is bound in the first predicate, but y is free. In the second predicate, y is bound but x is free.

All variables that occur in a propositional function must be bound to turn it into a proposition.

Thus, the left-hand side is a proposition, but the right-hand side is not. How can they be equivalent?

Notes