Master Theorem

When analyzing algorithms, recall that we only care about the asymptotic behavior.

Recursive algorithms are no different. Rather than solve exactly the recurrence relation associated with the cost of an algorithm, it is enough to give an asymptotic characterization.

The main tool for doing this is the master theorem.

Master Theorem II

Theorem (Master Theorem)

Let \( T(n) \) be a monotonically increasing function that satisfies
\[
T(n) = a T\left(\frac{n}{b}\right) + \Theta(n^d)
\]
where \( a \geq 1, b \geq 2, c > 0 \). If \( f(n) \in \Theta(n^d) \) where \( d \geq 0 \), then
\[
T(n) =
\begin{cases} 
  \Theta(n^d) & \text{if } a < b^d \\
  \Theta(n^d \log n) & \text{if } a = b^d \\
  \Theta(n^{\log_b a}) & \text{if } a > b^d 
\end{cases}
\]

Master Theorem

Example 1

Let \( T(n) = T\left(\frac{n}{2}\right) + \frac{1}{2}n^2 + n \). What are the parameters?
\[
a = 1 \\
b = 2 \\
d = 2
\]

Therefore which condition?
Since \( 1 < 2^2 \), case 1 applies.

Thus we conclude that
\[
T(n) \in \Theta(n^d) = \Theta(n^2)
\]
Master Theorem

Example 3

Let $T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1$. What are the parameters?

- $a = 3$
- $b = 2$
- $d = 1$

Therefore which condition?

Since $3 > 2^1$, case 3 applies. Thus we conclude that

$$T(n) \in \Theta(n^\log_b a) = \Theta(n^{\log_2 3})$$

Note that $\log_2 3 \approx 1.5849$. Can we say that $T(n) \in \Theta(n^{1.5849})$?

“Fourth” Condition

Recall that we cannot use the Master Theorem if $f(n)$ (the non-recursive cost) is not polynomial.

There is a limited 4-th condition of the Master Theorem that allows us to consider polylogarithmic functions.

Corollary

If $f(n) \in \Theta(n^{\log_b a} \log^k n)$ for some $k \geq 0$ then

$$T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$$

This final condition is fairly limited and we present it merely for completeness.

“Fourth” Condition

Example

Say that we have the following recurrence relation:

$$T(n) = 2T\left(\frac{n}{2}\right) + n \log n$$

Clearly, $a = 2$, $b = 2$ but $f(n)$ is not a polynomial. However,

$$f(n) \in \Theta(n \log n)$$

for $k = 1$, therefore, by the 4-th case of the Master Theorem we can say that

$$T(n) \in \Theta(n \log^2 n)$$