### Master Theorem

Slides by Christopher M. Bourke Instructor: Berthe Y. Choueiry

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Computer Science & Engineering 235 Introduction to Discrete Mathematics Section 7.3 of Rosen cse235@cse.unl.edu

### Master Theorem I

When analyzing algorithms, recall that we only care about the *asymptotic behavior*.

Recursive algorithms are no different. Rather than *solve* exactly the recurrence relation associated with the cost of an algorithm, it is enough to give an asymptotic characterization.

The main tool for doing this is the *master theorem*.

### Master Theorem II

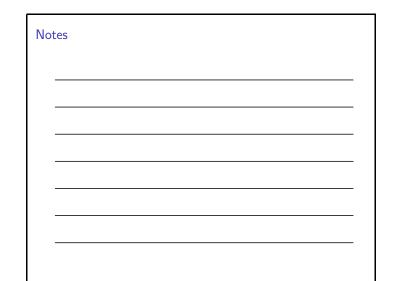
Theorem (Master Theorem)

Let  ${\cal T}(n)$  be a monotonically increasing function that satisfies

$$T(n) = aT(\frac{n}{b}) + f(n)$$
  
$$T(1) = c$$

where  $a \ge 1, b \ge 2, c > 0$ . If  $f(n) \in \Theta(n^d)$  where  $d \ge 0$ , then

 $T(n) = \left\{ \begin{array}{ll} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{array} \right.$ 



Notes



### Master Theorem Pitfalls

### You cannot use the Master Theorem if

- ▶ T(n) is not monotone, ex:  $T(n) = \sin n$
- f(n) is not a polynomial, ex:  $T(n) = 2T(\frac{n}{2}) + 2^n$
- b cannot be expressed as a constant, ex:  $T(n) = T(\sqrt{n})$

Note here, that the Master Theorem does not solve a recurrence relation.

Does the base case remain a concern?

Master Theorem Example 1

Let  $T(n) = T\left(\frac{n}{2}\right) + \frac{1}{2}n^2 + n$ . What are the parameters?

$$\begin{array}{rcl}
a &=& 1\\
b &=& 2\\
d &=& 2
\end{array}$$

Therefore which condition?

Since  $1 < 2^2$ , case 1 applies.

Thus we conclude that

$$T(n) \in \Theta(n^d) = \Theta(n^2)$$

# Master Theorem

Example 2

Let  $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n} + 42$ . What are the parameters?

$$\begin{array}{rrrr} a & = & 2 \\ b & = & 4 \\ d & = & \frac{1}{2} \end{array}$$

Therefore which condition?

Since  $2 = 4^{\frac{1}{2}}$ , case 2 applies.

Thus we conclude that

$$T(n) \in \Theta(n^d \log n) = \Theta(\sqrt{n} \log n)$$

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## Notes

### Master Theorem Example 3

Let  $T(n) = 3T\left(\frac{n}{2}\right) + \frac{3}{4}n + 1$ . What are the parameters?

$$\begin{array}{rrrrr} a & = & 3 \\ b & = & 2 \\ d & = & 1 \end{array}$$

Therefore which condition?

Since  $3 > 2^1$ , case 3 applies. Thus we conclude that

$$T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})$$

Note that  $\log_2 3 \approx 1.5849\ldots$  Can we say that  $T(n) \in \Theta(n^{1.5849})$  ?

## "Fourth" Condition

Recall that we cannot use the Master Theorem if f(n) (the non-recursive cost) is not polynomial.

There is a limited 4-th condition of the Master Theorem that allows us to consider polylogarithmic functions.

Corollary

If  $f(n) \in \Theta(n^{\log_b a} \log^k n)$  for some  $k \ge 0$  then

 $T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$ 

This final condition is fairly limited and we present it merely for completeness.

# "Fourth" Condition

Example

Say that we have the following recurrence relation:

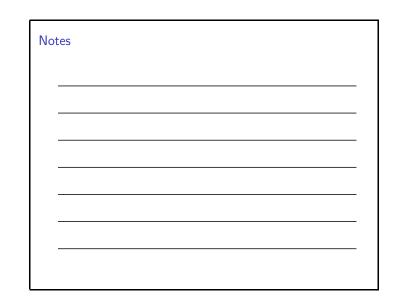
$$T(n) = 2T\left(\frac{n}{2}\right) + n\log n$$

Clearly,  $a=2, b=2 \mbox{ but } f(n)$  is not a polynomial. However,

$$f(n) \in \Theta(n \log n)$$

for k=1, therefore, by the 4-th case of the Master Theorem we can say that

 $T(n)\in \Theta(n\log^2 n)$ 





### Notes