Introduction to Logic

CSE235

Introduction to Logic

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Computer Science & Engineering 235
Introduction to Discrete Mathematics
Sections 1.1-1.2 of Rosen

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Introduction I

Propositional calculus (or logic) is the study of the logical relationship between objects called propositions and forms the basis of all mathematical reasoning and all automated reasoning.

Definition

A *proposition* is a statement that is either *true* or *false*, but not both (we usually denote a proposition by letters; $p, q, r, s, \ldots$).

Introduction II

Definition

The value of a proposition is called its *truth value*; denoted by T or 1 if it is true and F or 0 if it is false.

Opinions, interrogative and imperative sentences are not propositions.

Truth table:

p
0
1

Examples I

Example (Propositions)

- Today is Monday.
- The derivative of $\sin x$ is $\cos x$.
- Every even number has at least two factors.

Example (Not Propositions)

- C++ is the best language.
- When is the pretest?
- Do your homework.

Examples II

Example (Propositions?)

- $2 + 2 = 5$
- Every integer is divisible by 12.
- Microsoft is an excellent company.

Logical Connectives

Connectives are used to create a *compound* proposition from two or more other propositions.

- Negation (denoted $\neg$ or !)
- And (denoted $\wedge$) or Logical Conjunction
- Or (denoted $\vee$) or Logical Disjunction
- Exclusive Or (XOR, denoted $\oplus$)
- Implication (denoted $\rightarrow$)
- Biconditional; "if and only if" (denoted $\leftrightarrow$)

Negation

A proposition can be negated. This is also a proposition. We usually denote the negation of a proposition p by $\neg p$.

Example (Negated Propositions)

- Today is *not* Monday.
- *It is not the case that* today is Monday.
- *It is not the case that* the derivative of $\sin x$ is $\cos x$.

Truth table:

p	$\neg$p
0	1
1	0

Logical And

The logical connective AND is true only if *both* of the propositions are true. It is also referred to as a *conjunction*.

Example (Logical Connective: AND)

- It is raining and it is warm.
- $(2 + 3 = 5) \wedge (\sqrt{2} < 2)$
- Schrödinger's cat is dead and Schrödinger's cat is not dead.

Truth table:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Logical Or

The logical disjunction (or logical or) is true if one or both of the propositions are true.

Example (Logical Connective: OR)

- It is raining or it is the second day of lecture.
- $(2+2=5) \vee (\sqrt{2} < 2)$
- You may have cake or ice cream.[1]

Truth table:

p	q	$p \wedge q$	$p \vee q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

[1] Can I have both?

Exclusive Or

The exclusive or of two propositions is true when exactly *one* of its propositions is true and the other one is false.

Example (Logical Connective: Exclusive Or)

- The circuit is either is on or off.
- Let $ab < 0$, then either $a < 0$ or $b < 0$ but not both.
- You may have cake or ice cream, but not both.

Truth table:

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

Implications I

Definition

Let p and q be propositions. The implication

$$p \to q$$

is the proposition that is false when p is true and q is false and true otherwise.

Here, p is called the "hypothesis" (or "antecedent" or "premise") and q is called the "conclusion" or "consequence".

Truth table:

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Implications II

The implication $p \rightarrow q$ can be equivalently read as

- if p then q
- p implies q
- if p, q
- p only if q
- q if p
- q when p
- q whenever p
- p is a sufficient condition for q (p is sufficient for q)
- q is a necessary condition for p (q is necessary for p)
- q follows from p

Examples

Example

- If you buy your air ticket in advance, it is cheaper.
- If x is a real number, then $x^2 \geq 0$.
- If it rains, the grass gets wet.
- If the sprinklers operate, the grass gets wet.
- If $2 + 2 = 5$ then all unicorns are pink.

Exercise

Which of the following implications is true?

- If -1 is a positive number, then $2+2=5$.
- If -1 is a positive number, then $2+2=4$.
- If $\sin x = 0$ then $x = 0$.

Exercise

Which of the following implications is true?

- If -1 is a positive number, then $2 + 2 = 5$.
  true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.
- If -1 is a positive number, then $2 + 2 = 4$.
- If $\sin x = 0$ then $x = 0$.

Exercise

Which of the following implications is true?

- If -1 is a positive number, then $2+2=5$.
  true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.
- If -1 is a positive number, then $2+2=4$.
  true: for the same reason as above
- If $\sin x = 0$ then $x = 0$.

Exercise

Which of the following implications is true?

- If -1 is a positive number, then $2+2=5$.
  true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.
- If -1 is a positive number, then $2+2=4$.
  true: for the same reason as above
- If $\sin x = 0$ then $x = 0$.
  false: x can be any multiple of π; i.e. if we let $x = 2\pi$ then clearly $\sin x = 0$, but $x \neq 0$. The implication "if $\sin x = 0$ then $x = k\pi$ for some integer k" is true.

Biconditional

Definition

The *biconditional*

$$p \leftrightarrow q$$

is the proposition that is true when p and q have the same truth values. It is false otherwise.

Note that it is equivalent to $(p \to q) \wedge (q \to p)$

Truth table:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Examples

$p \leftrightarrow q$ can be equivalently read as

- p if and only if q
- p is necessary and sufficient for q
- if p then q, and conversely
- p iff q (Note typo in textbook, page 9, line 3.)

Example

- $x > 0$ if and only if x^2 is positive.
- The alarm goes off iff a burglar breaks in.
- You may have pudding if and only if you eat your meat.

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
- $x^2 \geq 0$ if and only if $x \geq 0$.

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
  true: both implications hold.
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
- $x^2 \geq 0$ if and only if $x \geq 0$.

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
  true: both implications hold.
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
  true: for the same reason above.
- $x^2 \geq 0$ if and only if $x \geq 0$.

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$
  true: both implications hold.
- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$
  true: for the same reason above.
- $x^2 \geq 0$ if and only if $x \geq 0$.
  false: The converse holds. That is, "if $x \geq 0$ then $x^2 \geq 0$". However, the implication is false; consider $x = -1$. Then the hypothesis is true, $(-1)^2 = 1^2 \geq 0$ but the conclusion fails.

Converse, Contrapositive, Inverse

Consider the proposition $p \rightarrow q$:

- Its *converse* is the proposistion $q \rightarrow p$.
- Its *inverse* is the proposistion $\neg p \rightarrow \neg q$.
- Its *contrapositive* is the proposistion $\neg q \rightarrow \neg p$.

Truth Tables I

Truth Tables are used to show the relationship between the truth values of individual propositions and the compound propositions based on them.

p	q	$p \wedge q$	$p \vee q$	p $\oplus$ q	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Table: Truth Table for Logical Conjunction, Disjunction, Exclusive Or, and Implication

Construct the Truth Table for the following compound proposition.

$$((p \wedge q) \vee \neg q)$$

p	q	$p \wedge q$	$\neg q$	$((p \wedge q) \vee \neg q)$
0	0			
0	1			
1	0			
1	1			

Constructing Truth Tables

Construct the Truth Table for the following compound proposition.

$$((p \wedge q) \vee \neg q)$$

p	q	$p \wedge q$	$\neg q$	$((p \wedge q) \vee \neg q)$
0	0	0		
0	1	0		
1	0	0		
1	1	1		

Constructing Truth Tables

Construct the Truth Table for the following compound proposition.

$$((p \land q) \lor \neg q)$$

p	q	$p \land q$	$\neg q$	$((p \land q) \lor \neg q)$
0	0	0	1	
0	1	0	0	
1	0	0	1	
1	1	1	0	

Constructing Truth Tables

Construct the Truth Table for the following compound proposition.

$$((p \land q) \lor \neg q)$$

p	q	$p \land q$	$\neg q$	$((p \land q) \lor \neg q)$
0	0	0	1	1
0	1	0	0	0
1	0	0	1	1
1	1	1	0	1

Precedence of Logical Operators

Just as in arithmetic, an ordering must be imposed on the use of logical operators in compound propositions.

Of course, parentheses can be used to make operators disambiguous:

$$\neg p \vee q \wedge \neg r \equiv (\neg p) \vee \big(q \wedge (\neg r)\big)$$

But to avoid using unnecessary parentheses, we define the following precedences:

1. $(\neg)$ Negation
2. $(\wedge)$ Conjunction
3. $(\vee)$ Disjunction
4. $(\rightarrow)$ Implication
5. $(\leftrightarrow)$ Biconditional

Usefulness of Logic

Logic is more precise than natural language:

- You may have cake or ice cream.
  Can I have both?
- If you buy your air ticket in advance, it is cheaper.
  Are there or not cheap last-minute tickets?

For this reason, logic is used for hardware and software *specification*.

Given a set of logic statements, one can decide whether or not they are satisfiable (i.e., consistent), although this is a costly process...

Bitwise Operations

Computers represent information as bits (*b*inary dig*its*).

A *bit string* is a sequence of bits, the length of the string is the number of bits in the string.

Logical connectives can be applied to bit strings (of equal length). To do this, we simply apply the connective rules to each bit of the string:

Example

0110	1010	1101	
0101	0010	1111	
0111	1010	1111	bitwise OR
0100	0010	1101	bitwise AND
0011	1000	0010	bitwise XOR

A *Boolean variable* is a variable that can have value 0 or 1.

What is SAT? SAT is the problem of determining whether or not a sentence in propositional logic (PL) is satisfiable. Characterizing SAT as an NP-complete problem is at the foundation of Theoretical Computer Science.

Defining SAT

- **Given**: a PL sentence.
- **Question**: Determine whether it is satisfiable or not.

What is a PL sentence? What does satisfiable mean?

- A sentence in PL is a *conjunction* of clauses
- A clause is a *disjunction* of literals
- A literal is a term or its negation
- A term is a (Boolean) variable (or proposition)

Example: $(a \vee b \vee \neg c \vee \neg d) \wedge (\neg b \vee c) \wedge (\neg a \vee c \vee d)$

A sentence in PL is a *satisfiable* iff we can assign truth value to the Boolean variables such that the sentence evaluates to true (i.e., holds).

Say you need to define a conditional statement as follows: "Increment x if all of the following conditions hold: $x > 0$, $x < 10$ and $x = 10$."

You may try:

```
if(0<x<10 OR x=10) x++;
```

But is not valid in C++ or Java. How can you modify this statement by using a logical equivalence?

Answer:

Say you need to define a conditional statement as follows: "Increment x if all of the following conditions hold: $x > 0$, $x < 10$ and $x = 10$."

You may try:

```
if(0<x<10 OR x=10) x++;
```

But is not valid in `C++` or Java. How can you modify this statement by using a logical equivalence?

Answer:

```
if(x>0 AND x<=10) x++;
```

Say we have the following loop:

```
while
  ((i<size AND A[i]>10) OR
   (i<size AND A[i]<0) OR
   (i<size AND (NOT (A[i]!= 0 AND NOT (A[i]>= 10))
```

Is this good code? Keep in mind:

- Readability.
- Extraneous code is inefficient and poor style.
- Complicated code is more prone to errors and difficult to debug.

Solution?

To manipulate a set of statements (here, logical propositions) for the sake mathematical argumentation, an important step is to replace one statement with another equivalent statement (i.e., with the same truth value).

Below, we discuss:

- Terminology
- Establising logical equivalences using truth tables
- Establising logical equivalences using known laws (of logical equivalences)

Definition

- A compound proposition that is always true, no matter what the truth values of the propositions that occur in it is called a *tautology*.
- A compound proposition that is always false is called a *contradiction*.
- Finally, a proposition that is neither a tautology nor a contradiction is called a *contingency*.

Example

A simple tautology is $p \vee \neg p$
A simple contradiction is $p \wedge \neg p$

Definition

Propositions p and q are *logically equivalent* if $p \leftrightarrow q$ is a tautology.

Informally, p and q are logically equivalent if whenever p is true, q is true, and vice versa.

Notation $p \equiv q$ ("p is equivalent to q"), $p \iff q$, $p \Leftrightarrow q$, $p \leftrightarrow q$.

Alort: $\equiv$ is **not** a logical connective.

Example

Are and $p \rightarrow q$ and $\neg p \lor q$ logically equivalent?

To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \lor q$
0	0			
0	1			
1	0			
1	1			

Example

Are and $p \rightarrow q$ and $\neg p \vee q$ logically equivalent?

To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1		
0	1	1		
1	0	0		
1	1	1		

Example

Are and $p \rightarrow q$ and $\neg p \vee q$ logically equivalent?

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p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1	1	
0	1	1	1	
1	0	0	0	
1	1	1	0	

Example

Are and $p \rightarrow q$ and $\neg p \vee q$ logically equivalent?

To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

Example

Are and $p \to q$ and $\neg p \vee q$ logically equivalent?

To find out, we construct the truth tables for each:

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

The two columns in the truth table are identical, thus we conclude that

$$p \to q \equiv \neg p \vee q$$

Another Example

(Exercise 25 from Rosen): Show that

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

p	q	r	$p \to r$	$(q \to r)$	$(p \to r) \lor (q \to r)$
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Another Example

(Exercise 25 from Rosen): Show that

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

p	q	r	$p \to r$	$(q \to r)$	$(p \to r) \vee (q \to r)$
0	0	0	1		
0	0	1	1		
0	1	0	1		
0	1	1	1		
1	0	0	0		
1	0	1	1		
1	1	0	0		
1	1	1	1		

Another Example

(Exercise 25 from Rosen): Show that

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

p	q	r	$p \to r$	$(q \to r)$	$(p \to r) \vee (q \to r)$
0	0	0	1	1	
0	0	1	1	1	
0	1	0	1	0	
0	1	1	1	1	
1	0	0	0	1	
1	0	1	1	1	
1	1	0	0	0	
1	1	1	1	1	

Another Example

(Exercise 25 from Rosen): Show that

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

p	q	r	$p \to r$	$(q \to r)$	$(p \to r) \vee (q \to r)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Now let's do it for $(p \wedge q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

Another Example
Continued

Now let's do it for $(p \wedge q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	0	
1	1	0	1	
1	1	1	1	

Another Example

Continued

Now let's do it for $(p \wedge q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

Another Example

Continued

Now let's do it for $(p \wedge q) \rightarrow r$:

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

The truth values are identical, so we conclude that the logical equivalence holds.

Tables of logical equivalences can be found in Rosen (page 24).

These and other can be found in a handout on the course web page `http://www.cse.unl.edu/~cse235/files/LogicalEquivalences.pdf`

Let's take a quick look at this Cheat Sheet

Logical equivalences can be used to construct additional logical equivalences.

Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$((p \wedge q) \rightarrow q) \iff \neg(p \wedge q) \vee q \qquad \text{Implication Law}$$

Using Logical Equivalences
Example 1

Logical equivalences can be used to construct additional logical equivalences.

Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$
\begin{array}{rcll}
((p \wedge q) \rightarrow q) & \Longleftrightarrow & \neg(p \wedge q) \vee q & \text{Implication Law} \\
 & \Longleftrightarrow & (\neg p \vee \neg q) \vee q & \text{De Morgan's Law (1st)}
\end{array}
$$

Logical equivalences can be used to construct additional logical equivalences.

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& \iff & (\neg p \vee \neg q) \vee q & \text{De Morgan's Law (1st)} \\
& \iff & \neg p \vee (\neg q \vee q) & \text{Associative Law}
\end{array}
$$

Using Logical Equivalences
Example 1

Logical equivalences can be used to construct additional logical equivalences.

Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$\begin{array}{rcll} ((p \wedge q) \rightarrow q) & \Longleftrightarrow & \neg(p \wedge q) \vee q & \text{Implication Law} \\ & \Longleftrightarrow & (\neg p \vee \neg q) \vee q & \text{De Morgan's Law (1st)} \\ & \Longleftrightarrow & \neg p \vee (\neg q \vee q) & \text{Associative Law} \\ & \Longleftrightarrow & \neg p \vee 1 & \text{Negation Law} \end{array}$$

Using Logical Equivalences

Example 1

Logical equivalences can be used to construct additional logical equivalences.

Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$
\begin{array}{rcll}
((p \wedge q) \rightarrow q) & \Longleftrightarrow & \neg(p \wedge q) \vee q & \text{Implication Law} \\
 & \Longleftrightarrow & (\neg p \vee \neg q) \vee q & \text{De Morgan's Law (1st)} \\
 & \Longleftrightarrow & \neg p \vee (\neg q \vee q) & \text{Associative Law} \\
 & \Longleftrightarrow & \neg p \vee 1 & \text{Negation Law} \\
 & \Longleftrightarrow & 1 & \text{Domination Law}
\end{array}
$$

Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition.
$(p \leftrightarrow \neg q)$

[1]See Table 8 (p25), but you are not allowed to use the table for the proof.

Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition.
$(p \leftrightarrow \neg q)$

$$\iff \quad (p \rightarrow \neg q) \wedge (\neg q \rightarrow p) \qquad \text{Equivalence Law}$$

[1] See Table 8 (p25), but you are not allowed to use the table for the proof.

Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

$\iff$	$(p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$	Equivalence Law
$\iff$	$(\neg p \vee \neg q) \wedge (q \vee p)$	Implication Law

[1] See Table 8 (p25), but you are not allowed to use the table for the proof.

Introduction to Logic

CSE235

Introduction

Usefulness of Logic

Propositional Equivalences

Using TT

Using L.E.

Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition.

$(p \leftrightarrow \neg q)$

$$\begin{array}{lll} \iff & (p \rightarrow \neg q) \wedge (\neg q \rightarrow p) & \text{Equivalence Law} \\ \iff & (\neg p \vee \neg q) \wedge (q \vee p) & \text{Implication Law} \\ \iff & \neg\left(\neg\left((\neg p \vee \neg q) \wedge (q \vee p)\right)\right) & \text{Double Negation} \end{array}$$

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Example (Exercise 17)[1]: Show that

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$$\begin{array}{lll}
\iff & (p \rightarrow \neg q) \wedge (\neg q \rightarrow p) & \text{Equivalence Law} \\
\iff & (\neg p \vee \neg q) \wedge (q \vee p) & \text{Implication Law} \\
\iff & \neg\left(\neg\left((\neg p \vee \neg q) \wedge (q \vee p)\right)\right) & \text{Double Negation} \\
\iff & \neg\left(\neg(\neg p \vee \neg q) \vee \neg(q \vee p)\right) & \text{De Morgan's Law}
\end{array}$$

[1] See Table 8 (p25), but you are not allowed to use the table for the proof.

Using Logical Equivalences

Example 2

Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition.
$(p \leftrightarrow \neg q)$

$$\begin{array}{lll}
\iff & (p \rightarrow \neg q) \wedge (\neg q \rightarrow p) & \text{Equivalence Law} \\
\iff & (\neg p \vee \neg q) \wedge (q \vee p) & \text{Implication Law} \\
\iff & \neg\left(\neg\left((\neg p \vee \neg q) \wedge (q \vee p)\right)\right) & \text{Double Negation} \\
\iff & \neg\left(\neg(\neg p \vee \neg q) \vee \neg(q \vee p)\right) & \text{De Morgan's Law} \\
\iff & \neg\left((p \wedge q) \vee (\neg q \wedge \neg p)\right) & \text{De Morgan's Law}
\end{array}$$

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Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

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$$\begin{array}{lll}
\iff & (p \rightarrow \neg q) \wedge (\neg q \rightarrow p) & \text{Equivalence Law} \\
\iff & (\neg p \vee \neg q) \wedge (q \vee p) & \text{Implication Law} \\
\iff & \neg\left(\neg\left((\neg p \vee \neg q) \wedge (q \vee p)\right)\right) & \text{Double Negation} \\
\iff & \neg\left(\neg(\neg p \vee \neg q) \vee \neg(q \vee p)\right) & \text{De Morgan's Law} \\
\iff & \neg\left((p \wedge q) \vee (\neg q \wedge \neg p)\right) & \text{De Morgan's Law} \\
\iff & \neg\left((p \wedge q) \vee (\neg p \wedge \neg q)\right) & \text{Commutative Law}
\end{array}$$

[1]See Table 8 (p25), but you are not allowed to use the table for the proof.

Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition. $(p \leftrightarrow \neg q)$

$\iff$	$(p \rightarrow \neg q) \wedge (\neg q \rightarrow p)$	Equivalence Law
$\iff$	$(\neg p \vee \neg q) \wedge (q \vee p)$	Implication Law
$\iff$	$\neg\left(\neg\left((\neg p \vee \neg q) \wedge (q \vee p)\right)\right)$	Double Negation
$\iff$	$\neg\left(\neg(\neg p \vee \neg q) \vee \neg(q \vee p)\right)$	De Morgan's Law
$\iff$	$\neg\left((p \wedge q) \vee (\neg q \wedge \neg p)\right)$	De Morgan's Law
$\iff$	$\neg\left((p \wedge q) \vee (\neg p \wedge \neg q)\right)$	Commutative Law
$\iff$	$\neg(p \leftrightarrow q)$	Equivalence Law
		(See Table 8, p25)

[1] See Table 8 (p25), but you are not allowed to use the table for the proof.

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$\neg(q \rightarrow p) \vee (p \wedge q)$

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$\neg(q \rightarrow p) \vee (p \wedge q)$

$$\iff \quad (\neg(\neg q \vee p)) \vee (p \wedge q) \quad \text{Implication Law}$$

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$\neg(q \rightarrow p) \vee (p \wedge q)$

$$\begin{array}{lll} \iff & (\neg(\neg q \vee p)) \vee (p \wedge q) & \text{Implication Law} \\ \iff & (q \wedge \neg p) \vee (p \wedge q) & \text{De Morgan's \& Double Negatio} \end{array}$$

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$\neg(q \rightarrow p) \vee (p \wedge q)$

$$\begin{array}{lll} \iff & (\neg(\neg q \vee p)) \vee (p \wedge q) & \text{Implication Law} \\ \iff & (q \wedge \neg p) \vee (p \wedge q) & \text{De Morgan's \& Double Negatio} \\ \iff & (q \wedge \neg p) \vee (q \wedge p) & \text{Commutative Law} \end{array}$$

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$$\neg(q \rightarrow p) \vee (p \wedge q)$$

$$\begin{array}{lll}
\iff & (\neg(\neg q \vee p)) \vee (p \wedge q) & \text{Implication Law} \\
\iff & (q \wedge \neg p) \vee (p \wedge q) & \text{De Morgan's \& Double Negatio} \\
\iff & (q \wedge \neg p) \vee (q \wedge p) & \text{Commutative Law} \\
\iff & q \wedge (\neg p \vee p) & \text{Distributive Law}
\end{array}$$

Using Logical Equivalences
Example 3

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$\neg(q \rightarrow p) \vee (p \wedge q)$

$\iff$	$(\neg(\neg q \vee p)) \vee (p \wedge q)$	Implication Law
$\iff$	$(q \wedge \neg p) \vee (p \wedge q)$	De Morgan's & Double Negatio
$\iff$	$(q \wedge \neg p) \vee (q \wedge p)$	Commutative Law
$\iff$	$q \wedge (\neg p \vee p)$	Distributive Law
$\iff$	$q \wedge 1$	Identity Law

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$\neg(q \rightarrow p) \vee (p \wedge q)$

$\iff$	$(\neg(\neg q \vee p)) \vee (p \wedge q)$	Implication Law
$\iff$	$(q \wedge \neg p) \vee (p \wedge q)$	De Morgan's & Double Negatio
$\iff$	$(q \wedge \neg p) \vee (q \wedge p)$	Commutative Law
$\iff$	$q \wedge (\neg p \vee p)$	Distributive Law
$\iff$	$q \wedge 1$	Identity Law
$\iff$	q	Identity Law

Recall the loop:

```
while((i<size AND A[i]>10) OR
      (i<size AND A[i]<0) OR
      (i<size AND (NOT (A[i]!= 0 AND NOT (A[i]>=
```

Now, using logical equivalences, simplify it.

Answer: Use De Morgan's Law and Distributivity.

```
while((i<size) AND
      ((A[i]>10 OR A[i]<0) OR
      (A[i]==0 OR A[i]>=10)))
```

Notice the ranges of all four conditions on A[i]; they can be merged and we can further simplify it to:

```
while((i<size) AND
      (A[i]>=10 OR A[i]<=0))
```

Programming Pitfall Note

In `C`, `C++` and Java, applying the commutative law is not such a good idea. These languages (compiler dependent) sometimes use "short-circuiting" for efficiency (at the machine level). For example, consider accessing an integer array `A` of size n.

```
if(i<n && A[i]==0) i++;
```

is not equivalent to

```
if(A[i]==0 && i<n) i++;
```