Functions

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Introduction

You’ve already encountered functions throughout your education.

\[ f(x, y) = x + y \]
\[ f(x) = x \]
\[ f(x) = \sin x \]

Here, however, we will study functions on discrete domains and ranges. Moreover, we generalize functions to mappings. Thus, there may not always be a "nice" way of writing functions like above.

Definition

A function \( f \) from a set \( A \) to a set \( B \) is an assignment of exactly one element of \( B \) to each element of \( A \). We write \( f(a) = b \) if \( b \) is the unique element of \( B \) assigned by the function \( f \) to the element \( a \in A \). If \( f \) is a function from \( A \) to \( B \), we write

\[ f : A \rightarrow B \]

This can be read as “\( f \) maps \( A \) to \( B \)”.

Note the subtlety:

- Each and every element in \( A \) has a single mapping.
- Each element in \( B \) may be mapped to by several elements in \( A \) or not at all.
Definitions
Terminology

**Definition**
Let \( f : A \rightarrow B \) and let \( f(a) = b \). Then we use the following terminology:

- \( A \) is the **domain** of \( f \), denoted \( \text{dom}(f) \).
- \( B \) is the **codomain** of \( f \).
- \( b \) is the **image** of \( a \).
- \( a \) is the **preimage** (antecedent) of \( b \).
- The **range** of \( f \) is the set of all images of elements of \( A \), denoted \( \text{rng}(f) \).

Definitions
Visualization

A function, \( f : A \rightarrow B \).

Definition I
More Definitions

**Definition**
Let \( f_1 \) and \( f_2 \) be functions from a set \( A \) to \( \mathbb{R} \). Then \( f_1 + f_2 \) and \( f_1 f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x) \\
(f_1 f_2)(x) = f_1(x) f_2(x)
\]

Example
Definition II
More Definitions

Let \( f_1(x) = x^4 + 2x^2 + 1 \) and \( f_2(x) = 2 - x^2 \) then

\[
(f_1 + f_2)(x) = (x^4 + 2x^2 + 1) + (2 - x^2) = x^4 + 4x^2 + 3
\]

\[
(f_1f_2)(x) = (x^4 + 2x^2 + 1) \cdot (2 - x^2) = -x^6 + 3x^2 + 2
\]

Definition

Let \( f : A \to B \) and let \( S \subseteq A \). The image of \( S \) is the subset of \( B \) that consists of all the images of the elements of \( S \). We denote the image of \( S \) by \( f(S) \), so that

\[
f(S) = \{f(s) | s \in S\}
\]

Definition III
More Definitions

Note that here, an image is a set rather than an element.

Example

Let

- \( A = \{a_1, a_2, a_3, a_4, a_5\} \)
- \( B = \{b_1, b_2, b_3, b_4\} \)
- \( f = \{(a_1, b_2), (a_2, b_3), (a_3, b_1), (a_4, b_1), (a_5, b_4)\} \)
- \( S = \{a_1, a_3\} \)

Draw a diagram for \( f \).

The image of \( S \) is \( f(S) = \{b_2, b_3\} \)

Definition IV
More Definitions

A function \( f \) whose domain and codomain are subsets of the set of real numbers is called strictly increasing if \( f(x) < f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \). A function \( f \) is called strictly decreasing if \( f(x) > f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \).
Injections, Surjections, Bijections I

Definitions

Definition
A function $f$ is said to be one-to-one (or injective) if
$$f(x) = f(y) \implies x = y$$
for all $x$ and $y$ in the domain of $f$. A function is an injection if it is one-to-one.

Intuitively, an injection simply means that each element in $B$ has at most one preimage (antecedent).

It may be useful to think of the contrapositive of this definition:
$$x \neq y \implies f(x) \neq f(y)$$

Injections, Surjections, Bijections II

Definitions

Definition
A function $f : A \to B$ is called onto (or surjective) if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a surjection if it is onto.

Again, intuitively, a surjection means that every element in the codomain is mapped. This implies that the range is the same as the codomain.

Injections, Surjections, Bijections III

Definitions

Definition
A function $f$ is a one-to-one correspondence (or a bijection, if it is both one-to-one and onto).

One-to-one correspondences are important because they endow a function with an inverse. They also allow us to have a concept of cardinality for infinite sets!

Let’s take a look at a few general examples to get the feel for these definitions.
Function Examples
A Non-function

\[
\begin{array}{|c|c|}
\hline
A & B \\
\hline \hline
a_1 & \bullet b_1 \\
\hline a_2 & \bullet b_2 \\
\hline a_3 & \bullet b_3 \\
\hline a_4 & \bullet b_4 \\
\hline
\end{array}
\]

This is not a function: Both \(a_1\) and \(a_2\) map to more than one element in \(B\).

Function Examples
A Function; Neither One-To-One Nor Onto

\[
\begin{array}{|c|c|}
\hline
A & B \\
\hline \hline
a_1 & \bullet b_1 \\
\hline a_2 & \bullet b_2 \\
\hline a_3 & \bullet b_3 \\
\hline a_4 & \bullet b_4 \\
\hline
\end{array}
\]

This function not one-to-one since \(a_1\) and \(a_3\) both map to \(b_1\). It is not onto either since \(b_4\) is not mapped to by any element in \(A\).

Function Examples
One-To-One, Not Onto

\[
\begin{array}{|c|c|}
\hline
A & B \\
\hline \hline
a_1 & \bullet b_1 \\
\hline a_2 & \bullet b_2 \\
\hline a_3 & \bullet b_3 \\
\hline a_4 & \bullet b_4 \\
\hline
\end{array}
\]

This function is one-to-one since every \(a_i \in A\) maps to a unique element in \(B\). However, it is not onto since \(b_4\) is not mapped to by any element in \(A\).
Function Examples
Onto, Not One-To-One

\[
\begin{array}{c|c}
A & B \\
\hline
a_1 & b_1 \\
 a_2 & b_2 \\
 a_3 & b_3 \\
 a_4 & b_3 \\
\end{array}
\]

This function is onto since every element \( b_i \in B \) is mapped to by some element in \( A \). However, it is not one-to-one since \( b_3 \) is mapped to more than one element in \( A \).

Function Examples
A Bijection

\[
\begin{array}{c|c}
A & B \\
\hline
a_1 & b_1 \\
 a_2 & b_2 \\
 a_3 & b_3 \\
 a_4 & b_4 \\
\end{array}
\]

This function is a bijection because it is both one-to-one and onto: every element in \( A \) maps to a unique element in \( B \) and every element in \( B \) is mapped by some element in \( A \).

Notes

Exercises I
Exercise I

Example

Let \( f : \mathbb{Z} \to \mathbb{Z} \) be defined by
\[
f(x) = 2x - 3
\]
What is the domain and range of \( f \)? Is it onto? One-to-one?

Clearly, \( \text{dom}(f) = \mathbb{Z} \). To see what the range is, note that
\[
b \in \text{rng}(f) \iff b = 2a - 3 \quad a \in \mathbb{Z}
\]
\[
\iff b = 2(a - 2) + 1 \quad \iff b \text{ is odd}
\]
Therefore, the range is the set of all odd integers. Since the range and codomain are different, \( \text{rng}(f) \neq \mathbb{Z} \) we can also conclude that \( f \) is not onto.

However, \( f \) is one-to-one. To prove this, note that

\[
\begin{align*}
    f(x_1) = f(x_2) & \implies 2x_1 - 3 = 2x_2 - 3 \\ 
    & \implies x_1 = x_2
\end{align*}
\]

follows from simple algebra.

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**Example**

Let \( f \) be as before, \( f(x) = 2x - 3 \)

but now define \( f : \mathbb{N} \to \mathbb{N} \). What is the domain and range of \( f \)? Is it onto? One-to-one?

By changing the domain/codomain in this example, \( f \) is not even a function anymore. Consider \( f(1) = 2 \cdot 1 - 3 = -1 \not\in \mathbb{N} \).
Therefore, any $x_1, x_2 \in \mathbb{Z}$ satisfies the equality (i.e. there are an infinite number of solutions). In particular $f(2) = f(3) = -1$.

It is also not onto. The function is a parabola with a global minimum (calculus exercise) at $\left( \frac{5}{2}, -\frac{25}{4} \right)$. Therefore, the function fails to map to any integer less than $-1$.

What would happen if we changed the domain/codomain?

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**Example**

Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = 2x^2 + 7x$$

Is this function one-to-one? Onto?

Again, since this is a parabola, it cannot be onto (where is the global minimum?).

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However, it is one-to-one. We follow a similar argument as before:

\[
\begin{align*}
\quad \quad f(x_1) = f(x_2) & \quad \Rightarrow \quad 2x_1^2 + 7x_1 = 2x_2^2 + 7x_2 \\
& \quad \Rightarrow \quad 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \\
& \quad \Rightarrow \quad (x_1 + x_2) = \frac{7}{2}
\end{align*}
\]

But $\frac{7}{2} \not\in \mathbb{Z}$ therefore, it must be the case that $x_1 = x_2$. It follows that $f$ is one-to-one.
Exercises I

Example

Define $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$f(x) = 3x^3 - x$$

Is $f$ one-to-one? Onto?

To see if its one-to-one, again suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{Z}$. Then

$$3x_1^3 - x_1 = 3x_2^3 - x_2 \Rightarrow 3(x_1^3 - x_2^3) = (x_1 - x_2)$$
$$\Rightarrow 3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = (x_1 - x_2)$$
$$\Rightarrow (x_1^2 + x_1x_2 + x_2^2) = \frac{1}{3}$$

Again, this is impossible since $x_1, x_2$ are integers, thus $f$ is one-to-one.

However, the function is not onto. Consider this counter example: $f(a) = 1$ for some integer $a$. If this were true, then it must be the case that

$$a(3a^2 - 1) = 1$$

Where $a$ and $(3a^2 - 1)$ are integers. But the only time we can ever get that the product of two integers is 1 is when we have $-1(-1)$ or $1(1)$ neither of which satisfy the equality.

Inverse Functions I

Definition

Let $f : A \rightarrow B$ be a bijection. The inverse function of $f$ is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of $f$ is denoted by $f^{-1}$. Thus $f^{-1}(b) = a$ when $f(a) = b$.

More succinctly, if an inverse exists,

$$f(a) = b \iff f^{-1}(b) = a$$
Inverse Functions II

Note that by the definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is invertible.

Why must a function be bijective to have an inverse?

- Consider the case where \( f \) is not one-to-one. This means that some element \( b \in B \) is mapped to by more than one element in \( A \), say \( a_1 \) and \( a_2 \). How can we define an inverse? Does \( f^{-1}(b) = a_1 \) or \( a_2 \)?
- Consider the case where \( f \) is not onto. This means that there is some element \( b \in B \) that is not mapped to by any \( a \in A \), therefore what is \( f^{-1}(b) \)?

Examples

Example I

Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
 f(x) = 2x - 3
\]

What is \( f^{-1} \)?

First, verify that \( f \) is a bijection (it is). To find an inverse, we use substitution:

- Let \( f^{-1}(y) = x \)
- Let \( y = 2x - 3 \) and solve for \( x \)
- Clearly, \( x = \frac{y+3}{2} \) so,
- \( f^{-1}(y) = \frac{y+3}{2} \).
Examples

Example II

Let \( f(x) = x^2 \)

What is \( f^{-1} \)?

No domain/codomain has been specified. Say \( f: \mathbb{R} \to \mathbb{R} \) is \( f \) a bijection? Does an inverse exist?

No, however if we specify that
\[
A = \{ x \in \mathbb{R} \mid x \leq 0 \}
\]

and
\[
B = \{ y \in \mathbb{R} \mid y \geq 0 \}
\]

then it becomes a bijection and thus has an inverse.

Examples

Example II Continued

To find the inverse, we again, let \( f^{-1}(y) = x \) and \( y = x^2 \). Solving for \( x \) we get \( x = \pm \sqrt{y} \). But which is it?

Since \( \text{dom}(f) \) is all nonpositive and \( \text{rng}(f) \) is nonnegative, \( y \) must be positive, thus
\[
f^{-1}(y) = -\sqrt{y}
\]

Thus, it should be clear that domains/codomains are just as important to a function as the definition of the function itself.

Examples

Example III

Let \( f(x) = 2^x \)

What should the domain/codomain be for this to be a bijection?

What is the inverse?

The function should be \( f: \mathbb{R} \to \mathbb{R}^+ \). What happens when we include 0? Restrict either one to \( \mathbb{Z} \)?

Let \( f^{-1}(y) = x \) and \( y = 2^x \), solving for \( x \) we get \( x = \log_2(y) \).

Therefore,
\[
f^{-1}(y) = \log_2(y)
\]
**Composition I**

The values of functions can be used as the input to other functions.

**Definition**

Let \( g : A \to B \) and let \( f : B \to C \). The composition of the functions \( f \) and \( g \) is

\[
(f \circ g)(x) = f(g(x))
\]

**Composition II**

Note the order that you apply a function matters—you go from inner most to outer most.

The composition \( f \circ g \) cannot be defined unless the range of \( g \) is a subset of the domain of \( f \);

\[
f \circ g \text{ is defined } \iff \text{rng}(g) \subseteq \text{dom}(f)
\]

It also follows that \( f \circ g \) is not necessarily the same as \( g \circ f \).

**Composition of Functions**

The composition of two functions.
Example

Let \( f \) and \( g \) be functions, \( \mathbb{R} \to \mathbb{R} \) defined by

\[
\begin{align*}
  f(x) &= 2x - 3 \\
  g(x) &= x^2 + 1
\end{align*}
\]

What are \( f \circ g \) and \( g \circ f \)?

Note that \( f \) is bijective, thus \( \text{dom}(f) = \text{rng}(f) = \mathbb{R} \). For \( g \), we have that \( \text{dom}(g) = \mathbb{R} \) but that \( \text{rng}(g) = \{ x \in \mathbb{R} \mid x \geq 1 \} \).

Composition

Example I

Even so, \( \text{rng}(g) \subseteq \text{dom}(f) \) and so \( f \circ g \) is defined. Also, \( \text{rng}(f) \subseteq \text{dom}(g) \) so \( g \circ f \) is defined as well.

\[
(f \circ g)(x) = g(f(x)) = g(2x - 3) = (2x - 3)^2 + 1 = 4x^2 - 12x + 10
\]

and

\[
(g \circ f)(x) = f(g(x)) = f(x^2 + 1) = 2(x^2 + 1) - 3 = 2x^2 - 1
\]

Equality

Though intuitive, we formally state what it means for two functions to be equal.

Lemma

Two functions \( f \) and \( g \) are equal if and only if \( \text{dom}(f) = \text{dom}(g) \) and

\[
\forall a \in \text{dom}(f) (f(a) = g(a))
\]
Associativity

Though the composition of functions is not commutative \((f \circ g \neq g \circ f)\), it is associative.

Lemma

Composition of functions is an associative operation; that is,
\[(f \circ g) \circ h = f \circ (g \circ h)\]

Important Functions

Identity Function

Definition

The identity function on a set \(A\) is the function
\(\iota: A \rightarrow A\)
defined by \(\iota(a) = a\) for all \(a \in A\). This symbol is the Greek letter iota.

One can view the identity function as a composition of a function and its inverse;
\(\iota(a) = (f \circ f^{-1})(a)\)

Moreover, the composition of any function \(f\) with the identity function is itself \(f\);
\[(f \circ \iota)(a) = (\iota \circ f)(a) = f(a)\]

Inverses & Identity

The identity function, along with the composition operation gives us another characterization for when a function has an inverse.

Theorem

Functions \(f: A \rightarrow B\) and \(g: B \rightarrow A\) are inverses if and only if
\[g \circ f = \iota_A\text{ and }f \circ g = \iota_B\]

That is,
\[\forall a \in A, b \in B((g(f(a)) = a \land f(g(b)) = b)\]
Important Functions I
Absolute Value Function

Definition
The absolute value function, denoted \(|x|\) is a function \(f : \mathbb{R} \rightarrow \{y \in \mathbb{R} \mid y \geq 0\}\). Its value is defined by

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

Floor & Ceiling Functions

Definition
The floor function, denoted \(\lfloor x \rfloor\) is a function \(\mathbb{R} \rightarrow \mathbb{Z}\). Its value is the largest integer that is less than or equal to \(x\).

The ceiling function, denoted \(\lceil x \rceil\) is a function \(\mathbb{R} \rightarrow \mathbb{Z}\). Its value is the smallest integer that is greater than or equal to \(x\).
Factorial Function

The factorial function gives us the number of permutations (that is, uniquely ordered arrangement) of a collection of \( n \) objects.

**Definition**

The *factorial function*, denoted \( n! \), is a function \( \mathbb{N} \rightarrow \mathbb{Z}^+ \). Its value is the product of the first \( n \) positive integers.

\[
n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n
\]

Notes

The factorial function is defined on a discrete domain. In many applications, it is useful to consider a continuous version of the function (say if we want to differentiate it).

To this end, we have *Stirling’s Formula*:

\[
n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n
\]