Asymptotics

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Introduction

Magnitude Graph

0 5 10 15 20
5
10
15
20
n
f(n)
f(n) = log ( n)
f(n) = x
f(n) = n log (n)
f(n) = n2
f(n) = n3
f(n) = 2 n
f(n) = n!

Big-O Definition

Definition
Let \( f \) and \( g \) be two functions \( f, g : \mathbb{N} \to \mathbb{R}^+ \). We say that
\[
f(n) \in O(g(n)) \\
\text{(read: } f \text{ is } O \text{ of } g \text{)}
\]
if there exists a constant \( c \in \mathbb{R}^+ \) and \( n_0 \in \mathbb{N} \) such that for every integer \( n \geq n_0 \),
\[
f(n) \leq cg(n)
\]

- Big-O is actually Omicron, but it suffices to write “O”
- Intuition: \( f \) is (asymptotically) less than or equal to \( g \)
- Big-O gives an asymptotic upper bound

Introduction I

Recall that we are really only interested in the Order of Growth of an algorithm’s complexity.

How well does the algorithm perform as the input size grows;
\[
n \to \infty
\]

We have seen how to mathematically evaluate the cost functions of algorithms with respect to their input size \( n \) and their elementary operation.

However, it suffices to simply measure a cost function’s asymptotic behavior.

Big-Omega Definition

Definition
Let \( f \) and \( g \) be two functions \( f, g : \mathbb{N} \to \mathbb{R}^+ \). We say that
\[
f(n) \in \Omega(g(n)) \\
\text{(read: } f \text{ is } \Omega \text{ of } g \text{)}
\]
if there exist \( c \in \mathbb{R}^+ \) and \( n_0 \in \mathbb{N} \) such that for every integer \( n \geq n_0 \),
\[
f(n) \geq cg(n)
\]

- Intuition: \( f \) is (asymptotically) greater than or equal to \( g \)
- Big-Omega gives an asymptotic lower bound

In practice, specific hardware, implementation, languages, etc. will greatly affect how the algorithm behaves. However, we want to study and analyze algorithms in and of themselves, independent of such factors.

For example, an algorithm that executes its elementary operation 10\( n \) times is better than one which executes it .0005\( n^2 \) times.
Moreover, algorithms that have running times \( n^2 \) and 2000\( n^2 \) are considered to be asymptotically equivalent.
Big-Theta Definition

Definition
Let \( f \) and \( g \) be two functions \( f, g : \mathbb{N} \rightarrow \mathbb{R}^+ \). We say that
\[
f(n) \in \Theta(g(n))
\]
(read: \( f \) is Big-Theta of \( g \)) if there exist constants \( c_1, c_2 \in \mathbb{R}^+ \) and \( n_0 \in \mathbb{N} \) such that for every integer \( n \geq n_0 \),
\[
c_1g(n) \leq f(n) \leq c_2g(n)
\]
▶ Intuition: \( f \) is (asymptotically) equal to \( g \).
▶ \( f \) is bounded above and below by \( g \).
▶ Big-Theta gives an asymptotic equivalence.

Asymptotic Properties I

Theorem
For \( f_1(n) \in \mathcal{O}(g_1(n)) \) and \( f_2 \in \mathcal{O}(g_2(n)) \),
\[
f_1(n) + f_2(n) \in \mathcal{O}(\max\{g_1(n), g_2(n)\})
\]
This property implies that we can ignore lower order terms. In particular, for any polynomial \( p(n) \) with degree \( k \), \( p(n) \in \mathcal{O}(n^k)^1 \)
In addition, this gives us justification for ignoring constant coefficients. That is, for any function \( f(n) \) and positive constant \( c \),
\[
f(n) \in \Theta(f(n))
\]

Asymptotic Proof Techniques

Definitional Proof - Example II
Example
Let \( f(n) = 2n^2 + n \) and \( g(n) = n^3 \). Find a tight bound of the form \( f(n) \in \Delta(g(n)) \).
Our intuition tells us that
\[
f(n) \in \mathcal{O}(n^3)
\]

Asymptotic Proof Techniques

Definitional Proof - Example I
Example
Let \( f(n) = 2n^2 + n \) and \( g(n) = n^3 \). Our intuition should tell us that \( f(n) \in \mathcal{O}(g(n)) \). Simply using the definition confirms this:
\[
21n^2 + n \leq cn^3
\]
holds for, say \( c = 3 \) and for all \( n \geq n_0 = 8 \) (in fact, an infinite number of pairs can satisfy this equation).

Asymptotic Proof Techniques

Definitional Proof - Example I
Example
Let \( f(n) = 2n^2 + n \) and \( g(n) = n^3 \). Find a tight bound of the form \( f(n) \in \Delta(g(n)) \).
Our intuition tells us that
\[
f(n) \in \mathcal{O}(n^3)
\]

Asymptotic Proof Techniques

Definitional Proof
Proving an asymptotic relationship between two given functions \( f(n) \) and \( g(n) \) can be done intuitively for most of the functions you will encounter, all polynomials for example. However, this does not suffice as a formal proof.
To prove a relationship of the form \( f(n) \in \Delta(g(n)) \), where \( \Delta \) is one of \( \mathcal{O}, \Omega, \Theta \), can be done simply using the definitions, that is:
▶ find a value for \( c \) (or \( c_1 \) and \( c_2 \)).
▶ find a value for \( n_0 \).
(But this is not the only way.)
Asymptotic Proof Techniques
Definitional Proof - Example II

Proof.

- If \( n \geq 1 \) it is clear that \( n \leq n^3 \) and \( n^2 \leq n^3 \).
- Therefore, we have that
  \[
  n^2 + n \leq n^3 + n^3 = 2n^3
  \]
- Thus, for \( n_0 = 1 \) and \( c = 2 \), by the definition of Big-O, we have that \( f(n) \in O(g(n)) \).

Asymptotic Proof Techniques
Definitional Proof - Example III

Example

Let \( f(n) = n^3 + 4n^2 \) and \( g(n) = n^2 \). Find a tight bound of the form \( f(n) \in \Delta(g(n)) \).

Here, our intuition should tell us that \( f(n) \in \Omega(g(n)) \).

Asymptotic Proof Techniques
Trick for polynomial of degree 2

If you have a polynomial of degree 2 such as \( an^2 + bn + c \), you can prove it is \( \Theta(n^2) \) using the following values:

- \( c_1 = \frac{a}{4} \)
- \( c_2 = \frac{7a}{4} \)
- \( n_0 = 2 \cdot \max(\frac{|b|}{a}, \sqrt{\frac{|c|}{a}}) \)

Limit Method

Now try this one:

\[
\begin{align*}
  f(n) &= n^{50} + 12n^3 \log^4 n - 1243n^{12} + 245n^6 \log n + 12\log^2 n - \log n \\
g(n) &= 12n^{50} + 24\log^4 n + 3 - \frac{\log n}{n^3} + 12
\end{align*}
\]

Using the formal definitions can be very tedious especially when one has very complex functions. It is much better to use the Limit Method which uses concepts from calculus.

Limit Method Process

Say we have functions \( f(n) \) and \( g(n) \). We set up a limit quotient between \( f \) and \( g \) as follows:

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 
  0 & \text{then } f(n) \in O(g(n)) \\
  c > 0 & \text{then } f(n) \in \Theta(g(n)) \\
  \infty & \text{then } f(n) \in \Omega(g(n))
\end{cases}
\]

- Justifications for the above can be proven using calculus, but for our purposes the limit method will be sufficient for showing asymptotic inclusions.
- Always try to look for algebraic simplifications first.
- If \( f \) and \( g \) both diverge or converge on zero or infinity, then you need to apply l’Hôpital’s Rule.
Using l’Hôpital’s Rule will
We set up our limit,
We prove using limits.
Using algebra,
Therefore we conclude that the quotient converges to zero
Now we use the following Theorem without proof:

\[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)} \]

Why do we have to use l’Hôpital’s Rule? Consider the following function:
\[ f(x) = \frac{\sin x}{x} \]
Clearly, \( \sin 0 = 0 \) so you may say that \( f(x) = 0 \). However, the denominator is also zero so you may say \( f(x) = \infty \), but both are wrong.

Observe the graph of \( f(x) \):

\[ \text{Figure: } f(x) = \sin x \]
Clearly, though \( f(x) \) is undefined at \( x = 0 \), the limit still exists.
Applying l’Hôpital’s Rule gives us the correct answer:
\[ \lim_{x \to 0} \frac{\sin x'}{x'} = \frac{\cos x}{1} = 1 \]

Limit Method
Example 1

Let \( f(n) = 2^n \), \( g(n) = 3^n \). Determine a tight inclusion of the form \( f(n) \in \Delta(g(n)) \).
What’s our intuition in this case?

Continued.
Using algebra,
\[ \lim_{n \to \infty} \frac{2^n}{3^n} = \left( \frac{2}{3} \right)^n \]
Now we use the following Theorem without proof:
\[ \lim_{n \to \infty} \alpha = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases} \]
Therefore we conclude that the quotient converges to zero thus,
\[ 2^n \in O(3^n) \]
Limit Method

Example 2 - Proof A

Proof.

We prove using limits.

We set up our limit,

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{\log_2 n}{\log_3 n^2}
\]

Here, we have to use the change of base formula for logarithms:

\[
\log_\alpha n = \frac{\log_\beta n}{\log_\beta \alpha}
\]

\[
\begin{align*}
\text{And we get that} & \quad \lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{\log_2 n}{\log_3 n^2} \\
\text{Here, we have to use the change of base formula for} & \quad \text{logarithms:} \\
\text{log}_\alpha n & \quad = \frac{\log_\beta n}{\log_\beta \alpha} \\
\end{align*}
\]

\[
\frac{\log_2 n}{2 \log_2 n} = \frac{\log_3 3}{2} \approx 0.7924 \ldots
\]

\[
\Rightarrow \text{So we conclude that } f(n) \in \Theta(g(n)).
\]

Limit Method

Example 2 - Proof B

Continued.

And we get that

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{\log_2 n}{\log_3 n^2} = \frac{\log_2 n}{2 \log_2 n} = \frac{\log_3 3}{2} \approx 0.7924 \ldots
\]

So we conclude that \( f(n) \in \Theta(g(n)). \)

Limit Properties

A useful property of limits is that the composition of functions is preserved.

Lemma

For the composition \( \circ \) of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then

\[
\lim_{n \to \infty} f_1(n) \circ \lim_{n \to \infty} f_2(n) = \lim_{n \to \infty} f_1(n) \circ f_2(n)
\]

Efficiency Classes

Some useful derivatives that you should memorize include

\[
\begin{align*}
\text{(n^k)'} & = kn^{k-1} \\
\text{(log}_\alpha n)' & = \frac{1}{n \log(\alpha)} \\
\text{(f}_1(n)f_2(n))' & = f_1'(n)f_2(n) + f_1(n)f_2'(n) \quad \text{(product rule)} \\
\text{(e^x)'} & = \ln(e)e^x \quad \text{Careful!}
\end{align*}
\]

Log Identities

\[
\begin{align*}
\text{Change of Base Formula: } & \quad \log_\alpha (n) = \frac{\log(n)}{\log(\alpha)} \\

\text{Log (n^k)} & = k \log(n) \\

\text{Log (ab)} & = \log(a) + \log(b)
\end{align*}
\]

Some Efficiency Classes

- Constant \( O(1) \)
- Logarithmic \( O(\log(n)) \)
- Linear \( O(n) \)
- Polylogarithmic \( O(\log^k(n)) \)
- Quadratic \( O(n^2) \)
- Cubic \( O(n^3) \)
- Polynomial \( O(n^k) \) for any \( k > 0 \)
- Exponential \( O(2^n) \)
- Super-Exponential \( O(2^{f(n)}) \) for \( f(n) = n^{1+\epsilon}, \epsilon > 0 \)

For example, \( n! \)
Asymptotics is easy, but remember:

▶ Always look for algebraic simplifications
▶ You must always give a rigorous proof
▶ Using the limit method is always the best
▶ Always show l’Hôpital’s Rule if need be
▶ Give as simple (and tight) expressions as possible