Asymptotics

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Introduction
Recall that we are really only interested in the Order of Growth of an algorithm’s complexity.
How well does the algorithm perform as the input size grows;

\[ n \rightarrow \infty \]

We have seen how to mathematically evaluate the cost functions of algorithms with respect to their input size \( n \) and their elementary operation.
However, it suffices to simply measure a cost function’s asymptotic behavior.

Magnitude Graph
\[
\begin{align*}
  f(n) &= \log(n) \\
  f(n) &= n \\
  f(n) &= n\log(n) \\
  f(n) &= n^2 \\
  f(n) &= n^3 \\
  f(n) &= 2^n \\
  f(n) &= n!
\end{align*}
\]
Introduction I

In practice, specific hardware, implementation, languages, etc. will greatly affect how the algorithm behaves. However, we want to study and analyze algorithms in and of themselves, independent of such factors.

For example, an algorithm that executes its elementary operation $10n$ times is better than one which executes it $0.005n^2$ times. Moreover, algorithms that have running times $n^2$ and $2000n^2$ are considered to be asymptotically equivalent.

Big-O Definition

Definition

Let $f$ and $g$ be two functions $f, g : N \rightarrow R^+$. We say that

$$f(n) \in O(g(n))$$

(read: $f$ is Big-"O" of $g$) if there exists a constant $c \in R^+$ and $n_0 \in N$ such that for every integer $n \geq n_0$,

$$f(n) \leq cg(n)$$

- Big-O is actually Omicron, but it suffices to write “O”
- Intuition: $f$ is (asymptotically) less than or equal to $g$
- Big-O gives an asymptotic upper bound

Notes

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Big-Omega Definition

Definition

Let $f$ and $g$ be two functions $f, g : N \rightarrow R^+$. We say that

$$f(n) \in \Omega(g(n))$$

(read: $f$ is Big-Omega of $g$) if there exist $c \in R^+$ and $n_0 \in N$ such that for every integer $n \geq n_0$,

$$f(n) \geq cg(n)$$

- Intuition: $f$ is (asymptotically) greater than or equal to $g$.
- Big-Omega gives an asymptotic lower bound

Notes

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Big-Theta Definition

Definition

Let \( f \) and \( g \) be two functions \( f, g : \mathbb{N} \to \mathbb{R}^+ \). We say that
\[ f(n) \in \Theta(g(n)) \]
(read: \( f \) is Big-Theta of \( g \)) if there exist constants \( c_1, c_2 \in \mathbb{R}^+ \) and \( n_0 \in \mathbb{N} \) such that for every integer \( n \geq n_0 \),
\[ c_1 g(n) \leq f(n) \leq c_2 g(n) \]

▶ Intuition: \( f \) is (asymptotically) equal to \( g \).
▶ \( f \) is bounded above and below by \( g \).
▶ Big-Theta gives an asymptotic equivalence.

Asymptotic Properties I

Theorem

For \( f_1(n) \in O(g_1(n)) \) and \( f_2 \in O(g_2(n)) \),
\[ f_1(n) + f_2(n) \in O(max\{g_1(n), g_2(n)\}) \]

This property implies that we can ignore lower order terms. In particular, for any polynomial \( p(n) \) with degree \( k \), \( p(n) \in O(n^k) \).
In addition, this gives us justification for ignoring constant coefficients. That is, for any function \( f(n) \) and positive constant \( c \),
\[ cf(n) \in \Theta(f(n)) \]

The proof is left as an exercise.

Asymptotic Properties II

Some obvious properties also follow from the definition.

Corollary

For positive functions, \( f(n) \) and \( g(n) \) the following hold:

▶ \( f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n)) \) and \( f(n) \in \Omega(g(n)) \)
▶ \( f(n) \in \Omega(g(n)) \iff g(n) \in \Omega(f(n)) \)

The proof is left as an exercise.

\(^1\)More accurately, \( p(n) \in \Theta(n^k) \)
Asymptotic Proof Techniques

Definitional Proof

Proving an asymptotic relationship between two given functions \( f(n) \) and \( g(n) \) can be done intuitively for most of the functions you will encounter; all polynomials for example. However, this does not suffice as a formal proof.

To prove a relationship of the form \( f(n) \in \Delta(g(n)) \) where \( \Delta \) is one of \( O, \Omega, \) or \( \Theta \), can be done simply using the definitions, that is:

▶ find a value for \( c \) (or \( c_1 \) and \( c_2 \)).
▶ find a value for \( n_0 \).

(But this is not the only way.)

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Example

Let \( f(n) = 21n^2 + n \) and \( g(n) = n^3 \). Our intuition should tell us that \( f(n) \in O(g(n)) \). Simply using the definition confirms this:

\[
21n^2 + n \leq cn^3
\]

holds for, say \( c = 3 \) and for all \( n \geq n_0 = 8 \) (in fact, an infinite number of pairs can satisfy this equation).

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Example

Let \( f(n) = n^2 + n \) and \( g(n) = n^3 \). Find a tight bound of the form \( f(n) \in \Delta(g(n)) \).

Our intuition tells us that

\[
f(n) \in O(n^3)
\]
Proof.
▶ If \( n \geq 1 \) it is clear that \( n \leq n^3 \) and \( n^2 \leq n^3 \).
▶ Therefore, we have that \( n^2 + n \leq n^3 + n^3 = 2n^3 \)
▶ Thus, for \( n_0 = 1 \) and \( c = 2 \), by the definition of Big-O, we have that \( f(n) \in O(g(n)) \).

Example
Let \( f(n) = n^3 + 4n^2 \) and \( g(n) = n^2 \). Find a tight bound of the form \( f(n) \in \Delta(g(n)) \).

Here, our intuition should tell us that
\[
f(n) \in \Omega(g(n))
\]
Asymptotic Proof Techniques

Trick for polynomial of degree 2

If you have a polynomial of degree 2 such as $an^2 + bn + c$, you can prove it is $\Theta(n^2)$ using the following values:

- $c_1 = \frac{a}{4}$
- $c_2 = \frac{b}{4}$
- $n_0 = 2 \cdot \max\left(\frac{|b|}{a}, \sqrt[4]{\frac{|c|}{a}}\right)$

Limit Method

Now try this one:

$$f(n) = n^{50} + 12n^3 \log_4 n - 1243n^2 + 245n^6 \log n + 12 \log 3 n$$

$$g(n) = 12n^{50} + 24 \log_{14} n^4 - \frac{\log n}{n^3} + 12$$

Using the formal definitions can be very tedious especially when one has very complex functions. It is much better to use the Limit Method which uses concepts from calculus.

Limit Method Process

Say we have functions $f(n)$ and $g(n)$. We set up a limit quotient between $f$ and $g$ as follows:

$$\lim_{n \to \infty} f(n) = \left\{ \begin{array}{ll} 0 & \text{then } f(n) \in O(g(n)) \\ c > 0 & \text{then } f(n) \in \Theta(g(n)) \\ \infty & \text{then } f(n) \in \Omega(g(n)) \end{array} \right.$$
I'Hôpital's Rule

Theorem

(I'Hôpital's Rule) Let \( f \) and \( g \), if the limit between the quotient \( \frac{f(n)}{g(n)} \) exists, it is equal to the limit of the derivative of the denominator and the numerator.

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}
\]

I'Hôpital's Rule I

Justification

Why do we have to use I'Hôpital's Rule? Consider the following function:

\[ f(x) = \frac{\sin x}{x} \]

Clearly, \( \sin 0 = 0 \) so you may say that \( f(x) = 0 \). However, the denominator is also zero so you may say \( f(x) = \infty \), but both are wrong.

I'Hôpital's Rule II

Justification

Observe the graph of \( f(x) \):

\[ f(x) = \frac{\sin x}{x} \]

Figure: \( f(x) = \frac{\sin x}{x} \)

Clearly, though \( f(x) \) is undefined at \( x = 0 \), the limit still exists. Applying I'Hôpital's Rule gives us the correct answer:

\[
\lim_{x \to 0} \frac{\sin x}{x} = \frac{\cos x}{1} = 1
\]
Example

Let \( f(n) = 2^n \), \( g(n) = 3^n \). Determine a tight inclusion of the form \( f(n) \in \Delta(g(n)) \).

What's our intuition in this case?

Proof.

- We prove using limits.
- We set up our limit,
  \[
  \lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{2^n}{3^n}
  \]
- Using l'Hôpital's Rule will get you no where:
  \[
  \frac{2^n'}{3^n'} = \frac{(\ln 2)2^n}{(\ln 3)3^n}
  \]
  Both numerator and denominator still diverge. We'll have to use an algebraic simplification.

Continued.

- Using algebra,
  \[
  \lim_{n \to \infty} \frac{2^n}{3^n} = \left( \frac{2}{3} \right)^n
  \]
- Now we use the following Theorem without proof:
  \[
  \lim_{n \to \infty} \alpha = \begin{cases} 
  0 & \text{if } \alpha < 1 \\
  1 & \text{if } \alpha = 1 \\
  \infty & \text{if } \alpha > 1 
  \end{cases}
  \]
- Therefore we conclude that the quotient converges to zero thus, 
  \[
  2^n \in O(3^n)
  \]
Example
Let \( f(n) = \log_2 n \), \( g(n) = \log_3 n^2 \). Determine a tight inclusion of the form \( f(n) \in \Delta(g(n)) \).

What's our intuition in this case?

Proof.

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{\log_2 n}{\log_3 n^2} = \frac{\log_2 n}{\log_3 n} = \frac{\log_2 2}{\log_3 3} = \frac{1}{2} \approx 0.7924 \ldots
\]

So we conclude that \( f(n) \in \Theta(g(n)) \).

Continued.

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \frac{\log_2 (n)}{\log_3 (n^2)} = \frac{\log_2 (n)}{2 \log_3 (n)} = \frac{\log_2 3}{2} \approx 0.7924 \ldots
\]

So we conclude that \( f(n) \in \Theta(g(n)) \).
Limit Properties

A useful property of limits is that the composition of functions is preserved.

Lemma

For the composition \( f_1 \circ f_2 \) of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then

\[
\lim_{n \to \infty} f_1(n) \circ \lim_{n \to \infty} f_2(n) = \lim_{n \to \infty} f_1(n) \circ f_2(n)
\]

Useful Identities & Derivatives

Some useful derivatives that you should memorize include

- \((n^k)’ = k n^{k-1}\)
- \((\log_b(n))’ = \frac{1}{n \ln b}\)
- \((f_1(n) f_2(n))’ = f_1’(n) f_2(n) + f_1(n) f_2’(n)\) (product rule)
- \((c^n)’ = \ln(c) c^n\) ← Careful!

Log Identities

- Change of Base Formula: \(\log_b(n) = \frac{\log_c(n)}{\log_c(b)}\)
- \(\log(n^k) = k \log(n)\)
- \(\log(ab) = \log(a) + \log(b)\)

Efficiency Classes

<table>
<thead>
<tr>
<th>Class</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>(O(1))</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>(O(\log(n)))</td>
</tr>
<tr>
<td>Linear</td>
<td>(O(n))</td>
</tr>
<tr>
<td>Polylogarithmic</td>
<td>(O(\log^k(n)))</td>
</tr>
<tr>
<td>Quadratic</td>
<td>(O(n^2))</td>
</tr>
<tr>
<td>Cubic</td>
<td>(O(n^3))</td>
</tr>
<tr>
<td>Polynomial</td>
<td>(O(n^k)) for any (k &gt; 0)</td>
</tr>
<tr>
<td>Exponential</td>
<td>(O(2^n))</td>
</tr>
<tr>
<td>Super-Exponential</td>
<td>(O(2^{f(n)})) for (f(n) = n^{1+\epsilon}, \epsilon &gt; 0)</td>
</tr>
</tbody>
</table>

For example, \(n!\)

Table: Some Efficiency Classes
Summary

Asymptotics is easy, but remember:

▶ Always look for algebraic simplifications
▶ You must always give a rigorous proof
▶ Using the limit method is always the best
▶ Always show l'Hôpital's Rule if need be
▶ Give as simple (and tight) expressions as possible