Title: Consistency Algorithms and operations on constraints

Material: Algorithms for Constraint Satisfaction Problems [Mackworth and Freuder AIJ'85]

New, revised, enhanced version

Foundations of Constraint Satisfaction

CSCE421/821, Fall 2001

`www.cse.unl.edu/~choueiry/F01-421-821/`

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Required reading:

Algorithms for Constraint Satisfaction Problems, Mackworth and Freuder AIJ'85

Recommended reading:

1– Consistency in Networks of Relations, Mackworth AIJ'77

2– Constraint Propagation with Interval Labels, Davis, AIJ'87

Upon demand:

Notes of Rina Dechter, Chapter 3. Confidential, available only to class

Outline

- Motivation and material
- Davis: Waltz algorithm, quick discussion
- Node and arc consistency and their complexity
- Path consistency and its complexity
- Some recent developments

Consistency checking

Motivation

CSP are solved with search (conditioning)

Search performance is affected by:

- problem size
- amount of backtracking

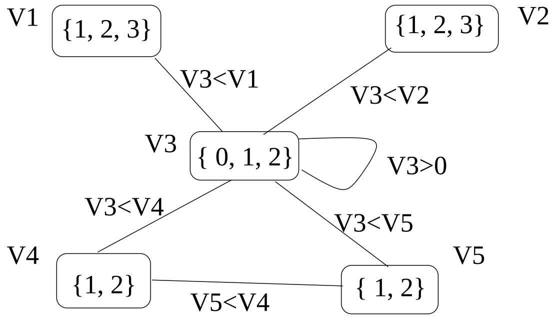
Backtracking may yield **thrashing**

Thrashing

exploring non-promising subtrees
and rediscovering the same inconsistent
combinations over and over again

$\longrightarrow$ Malady of backtrack search

(Some) causes of thrashing



Search order: V1, V2, V3, V4, V5

What happens in search if we:

- do not check the constraint on V3 (node inconsistency)
- do not check constraint between V3 and V5 (arc inconsistency)
- do not check constraints between V3–V4, V3–V5 and V4–V5 (path inconsistency)

Consistency checking

Goal: eliminate inconsistent combinations

Algorithms

Binary constraints:

- Node consistency
- Arc consistency (AC-1, AC-2, AC-3, ..., AC-7, AC-3.1, etc.)
- Path consistency (PC-1, PC-2, etc.)

Constraints of arbitrary arity:

- Waltz algorithm ancestor of AC's

Material

- Davis, AIJ'77: concentrates on the Waltz algorithm and studies its performance and quiescence depending on:
  - constraint types (order relations, bounded diff, algebraic, etc.)
  - domains types (continuous or finite)
- Mackworth, AIJ'77: presents NC, AC-1, AC-2, PC-1, PC-2
- Mackworth and Freuder, AIJ'85: studies their complexity

→ Mackworth and Freuder concentrate on finite domains
→ More people work on finite domains than continuous ones
→ Continuous domains can be quite tough

Constraint Propagation with Interval Labels

Ernest Davis

- 'Old' paper (1987): terminology slightly different

- Interval labels: continuous domains

  Section 8: sign labels (discrete)

- Concerned with Waltz algorithm

  (e.g., quiescence, completeness)

  Constraint types vs. domain types (Table 3)

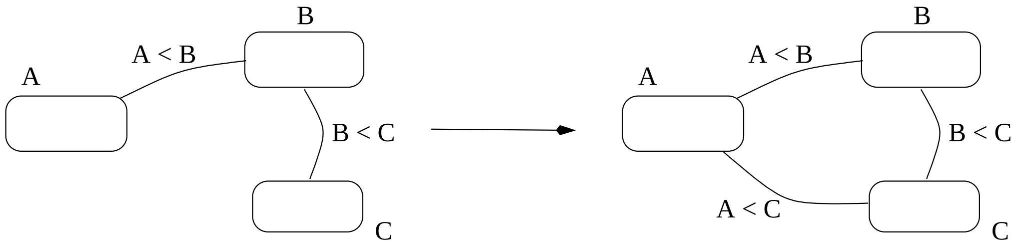
- Addresses applications from reasoning

  about Physical Systems (circuit, `SPAM`)

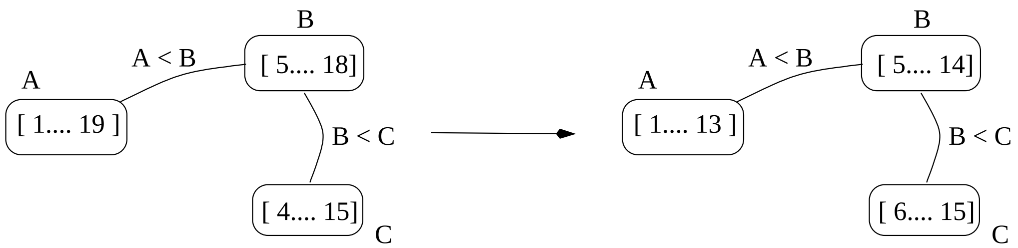
  about time relations (`TMM`)

$\longrightarrow$ Advice: read Section 3 at least (more if you wish)

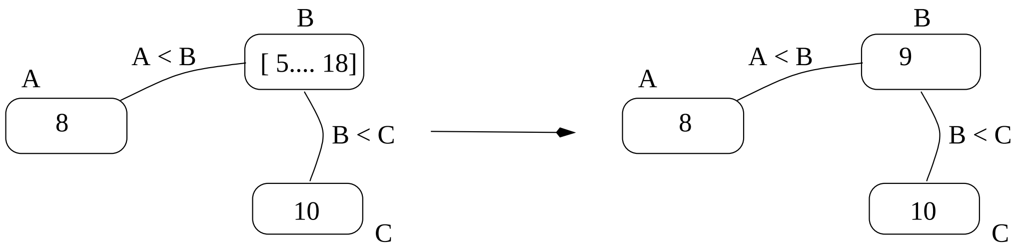
Constraint inference



Label inference

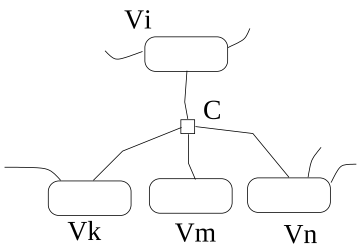


Value inference (dataflow constraint)



Waltz Algo. for label inference

`Refine`$(C(V_i, V_k, V_m, V_n), V_i)$ finds a new label for V_i consistent with C



`Revise`$(C(V_i, V_k, V_m, V_n))$ **refines** the domains of V_i, V_k, V_m, V_n by iterating over these variables until quiescence (no domains is further refined)

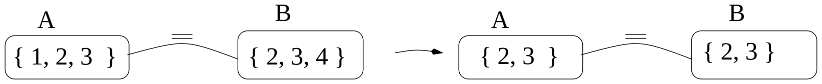
`Waltz Algorithm` **revises** all constraints by iterating over each constraint connected to a variable whose domain has been revised

`Waltz Algorithm` $\rightarrow$ ancestor of arc-consistency

WARNING

⟶ Completeness:

Running the `Waltz Algorithm` does not solve the problem



A=2 and B=3 is still not a solution!

⟶ Quiescence:

The Waltz algorithm may go into infinite loops even if problems is solvable

$$x \in [0, 100] \quad x = y$$
$$y \in [0, 100] \quad x = 2y$$

⟶ Davis provides a classification of problems f(constraint types, domain types) in terms of completeness and quiescence of the Waltz algorithm. (Table 3)

Importance of this paper

$\longrightarrow$ Establishes that constraints of bounded differences (temporal reasoning) can be efficiently solved by the Waltz algorithm ($O(n^3)$, n number of variables)

$\longrightarrow$ Early paper that attracts attention on the difficulty of label propagation in interval labels (continuous domains).

This work has been continued by Faltings (AIJ 92) who studied the early-quiescence problem of the Waltz algorithm.

Consistency checking algorithms in finite, binary CSPs and their complexity

Notation:

Given variables i, j, k with values x, y, z, the predicate

$$P_{ijk}(x, y, z) \text{ is true iff the 3-tuple } \langle x, y, z \rangle \in C_{ijk}$$

Node consistency: checking $P_i(x)$

Arc consistency: checking $P_{ij}(x, y)$

Path consistency: binary matrix manipulations

Worst-case complexity $\left\{ \begin{array}{c} \text{time} \\ \text{space} \end{array} \right\}$ as a function of the input parameters

<u>Upper bound:</u> $f(n)$ is $\mathcal{O}(g(n))$ means that $f(n) \leq c.g(n)$
f grows as g or slower

<u>Lower bound:</u> $f(n)$ is $\Omega(h(n))$ means that $f(n) \geq c.h(n)$
f grows as g or faster

Input parameters for a CSP:

$a =$ (max) size of a domain

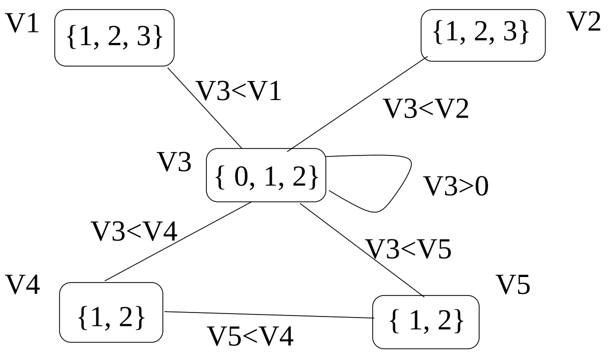
$d_k =$ degree of V_k ($\leq n - 1$)

$n =$ number of variables

$e =$ number of edges (or constraints) $\in [(n-1), n(n-1)/2]$

Node consistency (NC)

Procedure NC(i):
$D_i \leftarrow D_i \cap \{x | P_i(x)\}$
Begin
  for $i \leftarrow 1$ until n do NC(i)
end



Complexity of NC

Procedure NC(i):

$D_i \leftarrow D_i \cap \{x|P_i(x)\}$

Begin

for $i \leftarrow 1$ until n do NC(i)

end

For each variable, we check a values

We have n variables, we do $n.a$ checks

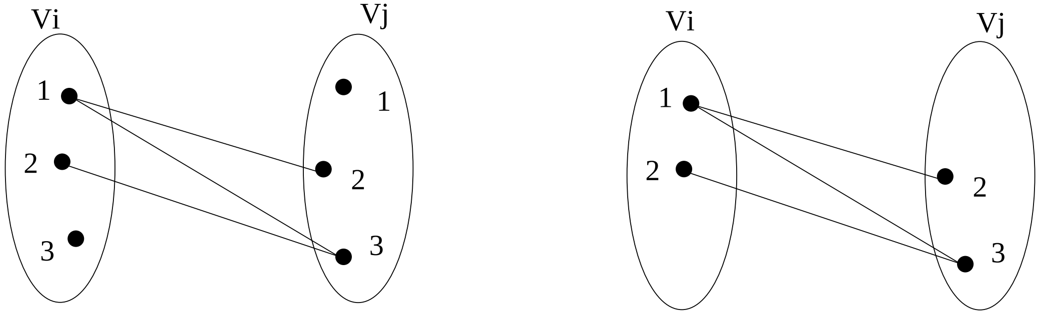
NC is $\mathcal{O}(a.n)$

Arc-consistency: Definition

Adapted from Rina Dechter

Given a constraint graph $\mathcal{G}$,

- A variable V_i is arc-consistent relative to V_j iff for every value a in D_{V_i}, there exists a value b in D_{V_j} such than $(a, b) \in C_{V_i,V_j}$.



- The constraint C_{V_i,V_j} is arc-consistent iff V_i is arc-consistent relative to V_j and V_j is arc-consistent relative to V_i.
- A binary CSP is arc-consistent iff every constraint (or subgraph of size 2) is arc-consistent

Procedure REVISE

Revises the domains of a variable i

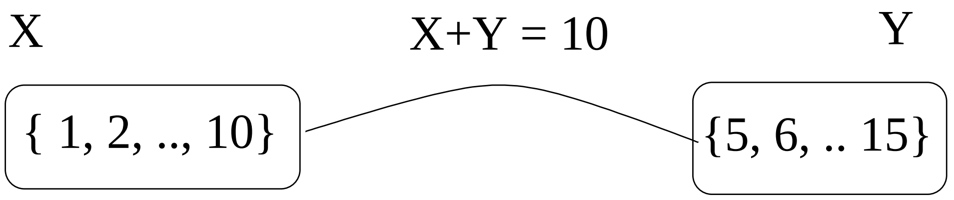
Procedure REVISE(i,j):
Begin
  DELETE $\leftarrow$ false
  for each $x \in D_i$ **do**
    if there is no $y \in D_j$ such that $P_{ij}(x, y)$ **then**
      begin
        delete x from D_i;
        DELETE $\leftarrow$ true
      end
  return DELETE
end

REVISE is directional

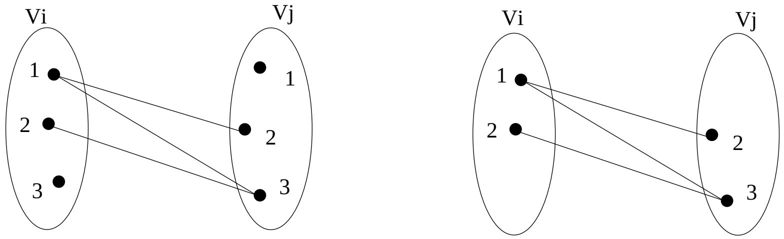
Complexity of REVISE? a^2

REVISE: Example *(courtesy of R. Dechter, UC-Irvine)*

Apply the REVISE procedure to the following example



Question: *Adapted from Rina Dechter*



Given two variables V_i and V_j, their domains D_{V_i} and D_{V_j}, and the constraint C_{V_i,V_j}, write the effect of the REVISE procedure as a sequence of operations in relational algebra

Hint: Think about the domain D_{V_i} as a unary constraint C_{V_i} and consider the composition of this unary constraint and the binary one..

Solution: $D_{V_i} \leftarrow D_{V_i} \cap \pi_{V_i}(C_{V_i,V_j} \bowtie D_{V_i})$
actually equivalent to $D_{V_i} \leftarrow \pi_{V_i}(C_{V_i,V_j} \bowtie D_{V_i})$

Arc consistency (AC-1)

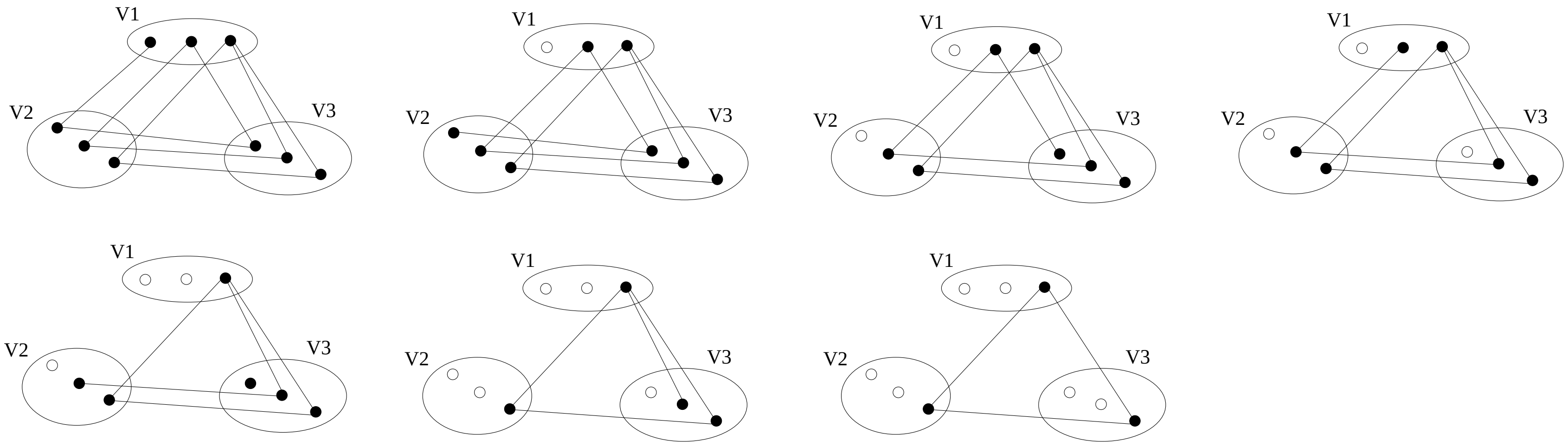
```
Procedure AC-1:
Begin
    for i ← 1 until n do NC(i)
    Q ← {(i, j)| (i, j) ∈ arcs(G), i ≠ j}
    repeat
        begin
        CHANGE ← false
        for each (i, j) ∈ Q do CHANGE ← (REVISE(i, j) or CHANCE)
        end
    until ¬ CHANGE
end
```

AC-1 does not update Q, the queue of arcs

Note: no algorithm can have time complexity below ea^2

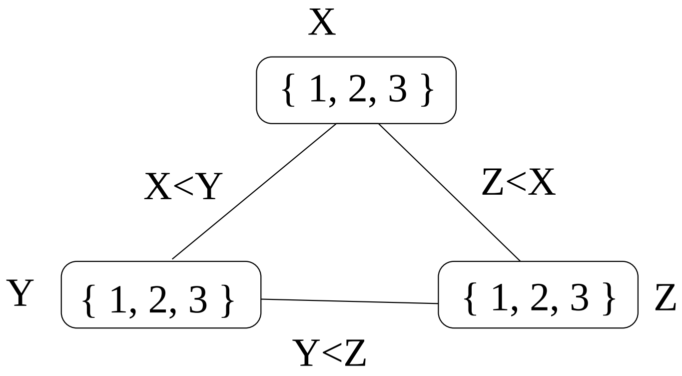
AC may discover the solution (I)

Example borrowed from Rina Dechter



AC may discover inconsistency (II)

Example borrowed from Rina Dechter



Complexity of AC-1

Procedure AC-1:
1 **Begin**
2 **for** $i \leftarrow 1$ **until** n **do** NC(i)
3 $Q \leftarrow \{(i, j) | \ (i, j) \in \text{arcs}(G), i \neq j$
4 **repeat**
5 **begin**
6 CHANGE $\leftarrow$ false
7 **for** each $(i, j) \in Q$ **do** CHANGE $\leftarrow$ (REVISE(i, j) **or** CHANCE)
8 end
9 **until** $\neg$ CHANGE
10 **end**

Q is not modified, and $|Q| = 2e$

4—9 repeats at most $n.a$ times

Each iteration has $|Q| = 2e$ calls to `REVISE`

`REVISE` requires at most a^2 checks of P_{ij}

$\longrightarrow$ AC-1 is $\mathcal{O}(a^3.n.e)$

AC-1 does not update Q, the queue of arcs

AC-2 iterates over arcs connected to at least one
node whose domains has been modified.
Nodes are ordered.

AC-3 same as AC-2, nodes are not ordered

Arc consistency (AC-3)

AC-3 iterates over arcs connected to at least one node whose domains has been modified

Procedure AC-3:

Begin

  for $i \leftarrow 1$ **until** n **do** NC(i)

  $Q \leftarrow \{(i,j) |\ (i,j) \in \text{arcs}(G), i \neq j$

  While Q is not empty **do**

    begin

    select and delete any arc (k,m) from Q

    if REVISE(k,m) **then** $Q \leftarrow Q \cup \{(i,k) | (i,k) \in \text{arcs}(G), i \neq k, i \neq m\}$

    end

end

Complexity of AC-3

Procedure AC-3:
1 **Begin**
2   **for** $i \leftarrow 1$ **until** n **do** NC(i)
3   $Q \leftarrow \{(i,j) |\ (i,j) \in \text{arcs}(G), i \neq j$
4   **While** Q is not empty **do**
5   **begin**
6   select and delete any arc (k,m) from Q
7   **if** REVISE(k,m) **then** $Q \leftarrow Q \cup \{(i,k) | (i,k) \in \text{arcs}(G), i \neq k, i \neq m\}$
8   **end**
9 **end**

First $|Q| = 2e$, then it grows and shrinks 4—8

Worst case:
$$\begin{cases} \text{1 element is deleted } D_{V_k} \text{ per iteration} \\ \text{none of the arcs added is in } Q \end{cases}$$

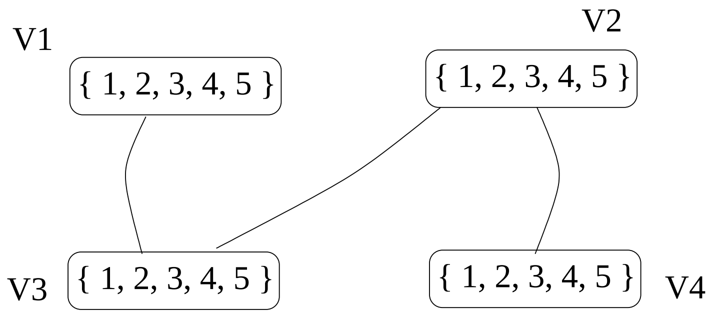
Line 7: $a.(d_k - 1)$
Lines 4–8: $2e + \sum_{k=1}^{n} a.(d_k - 1) = 2e + a.(2e - n)$
`REVISE`: a^2 checks of P_{ij}

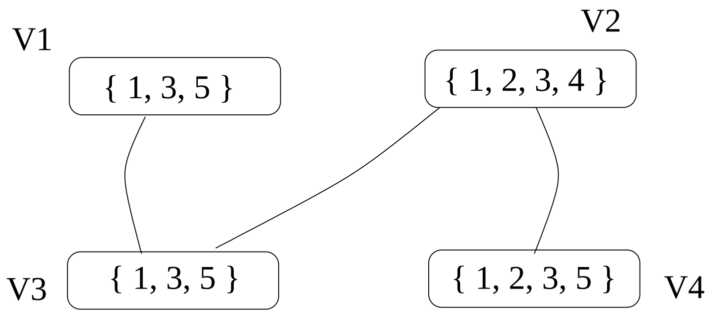
$\longrightarrow$ AC-3 is $\mathcal{O}(a^2(2e + a(2e - n)))$
Connected graph $\Rightarrow e \geq n - 1$, AC-3 is $\mathcal{O}(a^3 e)$

If AC-p, AC-3 is $\mathcal{O}(a^2 e) \Rightarrow$ AC-3 is $\Omega(a^2 e)$
Complete graph: $\mathcal{O}(a^3 n^2)$

Example: Apply AC-3

Thanks to Xu Lin





$$
\begin{aligned}
D_{V_1} &= \{1,2,3,4,5\} \\
D_{V_2} &= \{1,2,3,4,5\} \\
D_{V_3} &= \{1,2,3,4,5\} \\
D_{V_4} &= \{1,2,3,4,5\} \\
C_{V_2,V_3} &= \{(2,2),(4,5),(2,5),(3,5),(2,3),(5,1),(1,2),(5,3),(2,1),(1,1)\} \\
C_{V_1,V_3} &= \{(5,5),(2,4),(3,5),(3,3),(5,3),(4,4),(5,4),(3,4),(1,1),(3,1)\} \\
C_{V_2,V_4} &= \{(1,2),(3,2),(3,1),(4,5),(2,3),(4,1),(1,1),(4,3),(2,2),(1,5)\}
\end{aligned}
$$

Applying AC-3 *Thanks to Xu Lin*

- Queue = $\{C_{V_2,V_4}, C_{V_4,V_2}, C_{V_1,V_3}, C_{V_2,V_3}, C_{V_3,V_1}, C_{V_3,V_2}\}$
- Revise(V_2, V_4): $D_{V_2} \leftarrow D_{V_2} \setminus \{5\} = \{1,2,3,4\}$
- Queue = $\{C_{V_4,V_2}, C_{V_1,V_3}, C_{V_2,V_3}, C_{V_3,V_1}, C_{V_3,V_2}\}$
- Revise(V_4, V_2): $D_{V_4} \leftarrow D_{V_4} \setminus \{4\} = \{1,2,3,5\}$
- Queue = $\{C_{V_1,V_3}, C_{V_2,V_3}, C_{V_3,V_1}, C_{V_3,V_2}\}$
- Revise(V_1, V_3): $D_{V_1} \leftarrow \{1,2,3,4,5\}$
- Queue = $\{C_{V_2,V_3}, C_{V_3,V_1}, C_{V_3,V_2}\}$
- Revise(V_2, V_3): $D_{V_2} \leftarrow \{1,2,3,4\}$
- Queue = $\{C_{V_3,V_1}, C_{V_3,V_2}\}$
- Revise(V_3, V_1): $D_{V_3} \leftarrow D_{V_3} \setminus \{2\} = \{1,3,4,5\}$
- Queue = $\{C_{V_2,V_3}, C_{V_3,V_2}\}$
- Revise(V_2, V_3): $D_{V_2} \leftarrow D_{V_2}$
- Queue = $\{C_{V_3,V_2}\}$
- Revise(V_3, V_2): $D_{V_3} \leftarrow \{1,3,5\}$
- Queue = $\{C_{V_1,V_3}\}$
- Revise(V_1, V_3): $D_{V_1} \leftarrow \{1,3,5\}$

Main improvements

Thanks to Xu Lin

Mohr & Henderson (AIJ 86): AC-3 $\mathcal{O}(a^3e) \longrightarrow$ AC-4 $\mathcal{O}(a^2e)$
AC-4 is optimal

Trades repetition of consistency-check operations with heavy bookkeeping on which and how many values support a given value for a given variable;

- m: values that are active, not filtered
- s: lists all vvp's that support a given vvp
- *counter*: given a vvp, provides the number of support provided by a given variable

1. Generates data structures
2. Prepares data structures
3. Iterates over constraints

Step 1: Three data-structures are generated: m, s, and $counter$.

m	
$(V_4, 5)$	T
$(V_3, 2)$	T
$(V_3, 3)$	T
$(V_2, 2)$	T
$(V_3, 1)$	T
$(V_2, 3)$	T
$(V_1, 2)$	T
$(V_2, 1)$	T
$(V_1, 3)$	T
$(V_3, 4)$	T
$(V_4, 2)$	T
$(V_3, 5)$	T
$(V_4, 3)$	T
$(V_1, 1)$	T
$(V_2, 4)$	T
$(V_2, 5)$	T
$(V_4, 1)$	T
$(V_1, 4)$	T
$(V_1, 5)$	T
$(V_4, 4)$	T

s	
$(V_4, 5)$	NIL
$(V_3, 2)$	NIL
$(V_3, 3)$	NIL
$(V_2, 2)$	NIL
$(V_3, 1)$	NIL
$(V_2, 3)$	NIL
$(V_1, 2)$	NIL
$(V_2, 1)$	NIL
$(V_1, 3)$	NIL
$(V_3, 4)$	NIL
$(V_4, 2)$	NIL
$(V_3, 5)$	NIL
$(V_4, 3)$	NIL
$(V_1, 1)$	NIL
$(V_2, 4)$	NIL
$(V_2, 5)$	NIL
$(V_4, 1)$	NIL
$(V_1, 4)$	NIL
$(V_1, 5)$	NIL
$(V_4, 4)$	NIL

$counter$	
$(V_4, V_2), 3$	0
$(V_4, V_2), 2$	0
$(V_4, V_2), 1$	0
$(V_2, V_3), 1$	0
$(V_2, V_3), 2$	0
$(V_2, V_3), 3$	0
$(V_4, V_2), 5$	0
$(V_2, V_4), 1$	0
$(V_2, V_3), 4$	0
$(V_4, V_2), 4$	0
$(V_2, V_4), 2$	0
$(V_2, V_3), 5$	0
$(V_2, V_4), 3$	0
$(V_2, V_4), 4$	0
$(V_2, V_4), 5$	0
$(V_3, V_1), 1$	0
$(V_3, V_1), 2$	0
$(V_3, V_1), 3$	0
$(V_3, V_2), 1$	0
$(V_3, V_1), 4$	0
$(V_3, V_2), 2$	0
$(V_3, V_1), 5$	0
$(V_1, V_3), 1$	0
$(V_3, V_2), 3$	0
$(V_3, V_2), 4$	0
$(V_1, V_3), 3$	0
$(V_3, V_2), 5$	0
$(V_1, V_3), 2$	0
$(V_1, V_3), 5$	0
$(V_1, V_3), 4$	0

Step 2: Prepare data-structures s, and *counter*.

Checks for every constraint C_{V_i,V_j} all tuples $(V_i = a_i, V_j = b_j)$
When the tuple is allowed, then update:

$$s(V_j, b_j) \leftarrow s(V_j, b_j) + (V_i, a_i) \text{ and}$$
$$counter(V_j, b_j) \leftarrow counter(V_j, b_j) + 1$$

```
Update counter ((V_2, V_3), 2) to value 1
Update counter ((V_3, V_2), 2) to value 1
Update s-htable (V_2, 2) to value ((V_3, 2))
Update s-htable (V_3, 2) to value ((V_2, 2))

Update counter ((V_2, V_3), 4) to value 1
Update counter ((V_3, V_2), 5) to value 1
Update s-htable (V_2, 4) to value ((V_3, 5))
Update s-htable (V_3, 5) to value ((V_2, 4))

...
```

Consider the two constraints C_{V_2,V_3} and C_{V_3,V_2}.

$counter((V_2, V_3), 2)$ is 4 means that in the value 2 for V_2 is supported by 4 values for V_3.

s	
$(V_4, 5)$	NIL
$(V_3, 2)$	$((V_2, 1), (V_2, 2))$
$(V_3, 3)$	$((V_2, 5), (V_2, 2))$
$(V_2, 2)$	$((V_3, 1), (V_3, 3), (V_3, 5), (V_3, 2))$
$(V_3, 1)$	$((V_2, 1), (V_2, 2), (V_2, 5))$
$(V_2, 3)$	$((V_3, 5))$
$(V_1, 2)$	NIL
$(V_2, 1)$	$((V_3, 1), (V_3, 2))$
$(V_1, 3)$	NIL
$(V_3, 4)$	NIL
$(V_4, 2)$	NIL
$(V_3, 5)$	$((V_2, 3), (V_2, 2), (V_2, 4))$
$(V_4, 3)$	NIL
$(V_1, 1)$	NIL
$(V_2, 4)$	$((V_3, 5))$
$(V_2, 5)$	$((V_3, 3), (V_3, 1))$
$(V_4, 1)$	NIL
$(V_1, 4)$	NIL
$(V_1, 5)$	NIL
$(V_4, 4)$	NIL

counter	
$(V_4, V_2), 3$	0
$(V_4, V_2), 2$	0
$(V_4, V_2), 1$	0
$(V_2, V_3), 1$	2
$(V_2, V_3), 2$	4
$(V_2, V_3), 3$	1
$(V_4, V_2), 5$	0
$(V_2, V_4), 1$	0
$(V_2, V_3), 4$	1
$(V_4, V_2), 4$	0
$(V_2, V_4), 2$	0
$(V_2, V_3), 5$	2
$(V_2, V_4), 3$	0
$(V_2, V_4), 4$	0
$(V_2, V_4), 5$	0
$(V_3, V_1), 1$	0
$(V_3, V_1), 2$	0
$(V_3, V_1), 3$	0
$(V_3, V_2), 1$	3
$(V_3, V_1), 4$	0
$(V_3, V_2), 2$	2
$(V_3, V_1), 5$	0
$(V_1, V_3), 1$	0
$(V_3, V_2), 3$	2
$(V_3, V_2), 4$	0
$(V_1, V_3), 3$	0
$(V_3, V_2), 5$	3
$(V_1, V_3), 2$	0
$(V_1, V_3), 5$	0
$(V_1, V_3), 4$	0

Note that $counter[(V_3, V_2), 4] = 0$,

we need to remove 4 from domain of V_3

and update m

Since we remove $(V_3, 4)$, then for every element in (V_k, l) in $s[(V_3, y)]$ we decrement the $counter(V_k, V_3), l)$ by 1. (not shown here)

Eventually:

m	
$(V_4, 5)$	T
$(V_3, 2)$	NIL
$(V_3, 3)$	T
$(V_2, 2)$	T
$(V_3, 1)$	T
$(V_2, 3)$	T
$(V_1, 2)$	NIL
$(V_2, 1)$	T
$(V_1, 3)$	T
$(V_3, 4)$	NIL
$(V_4, 2)$	T
$(V_3, 5)$	T
$(V_4, 3)$	T
$(V_1, 1)$	T
$(V_2, 4)$	T
$(V_2, 5)$	NIL
$(V_4, 1)$	T
$(V_1, 4)$	NIL
$(V_1, 5)$	T
$(V_4, 4)$	NIL

Summary of arc-consistency algorithms

- **Mohr & Henderson** (AIJ 86): AC-4 is optimal
- Warning: Worst-case complexity is pessimistic.
  Better worst-case complexity: AC-4.
  Better average behavior: AC-3
- AC-5: specific to special constraints
- AC-6, AC-7: general but rely heavily on data structures for bookkeeping
- Now, back to AC-3: AC-2000, AC-2001, and AC-3.1
- Non-binary constraints: GAC

CSP parameters $\langle n, a, t, d \rangle$

- n is number of variables
- a is maximum domain size
- t is constraint tightness: $t = \frac{|forbidden\ tuples|}{|all\ tuples|}$
- d is constraint density $d = \frac{e - e_{min}}{e_{max} - e_{min}}$
  where e is the number of constraints,
  $e_{min} = (n-1)$, and
  $e_{max} = \frac{n(n-1)}{2}$

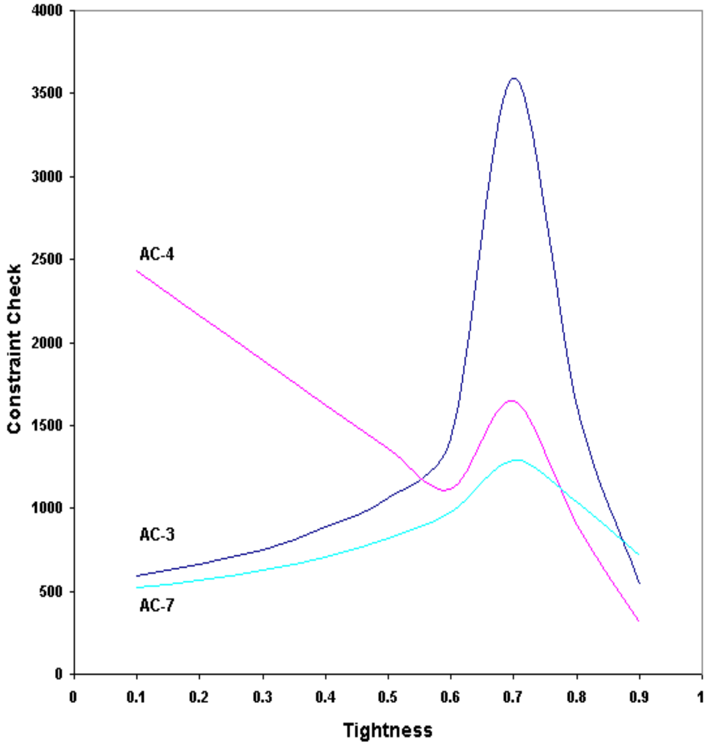
$\longrightarrow$ Constraints in random problem often generated uniform

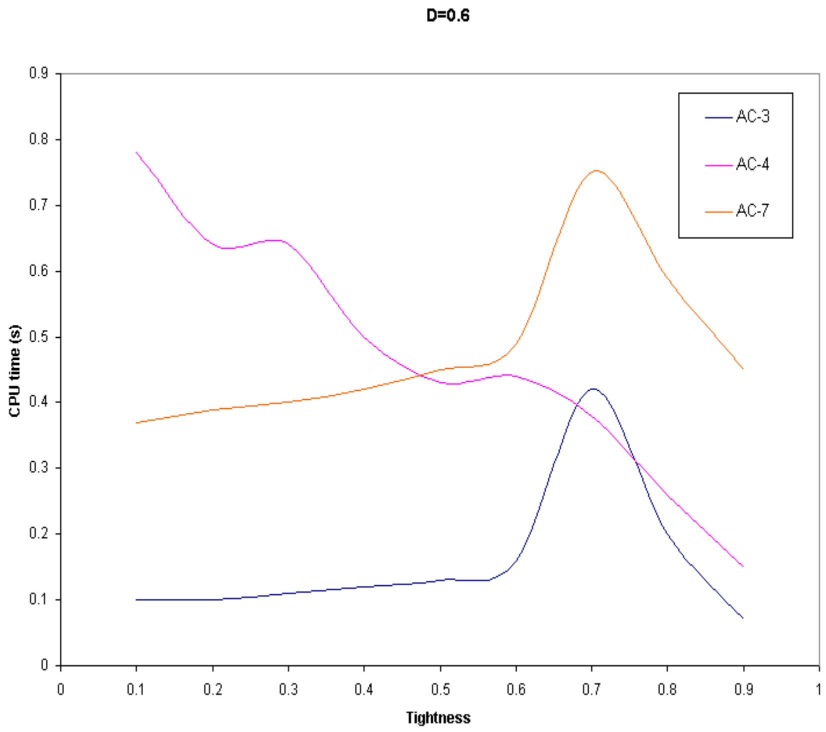
Criteria for performance comparison

- Worst-case, average-case complexity, time and space (theoretical)
- Number of constraint checks (theoretical, empirical)
- CPU time (empirical)

Performance comparison

Courtesy of Lin XU





Wisdom (?) *Free adaptation from Bessière*

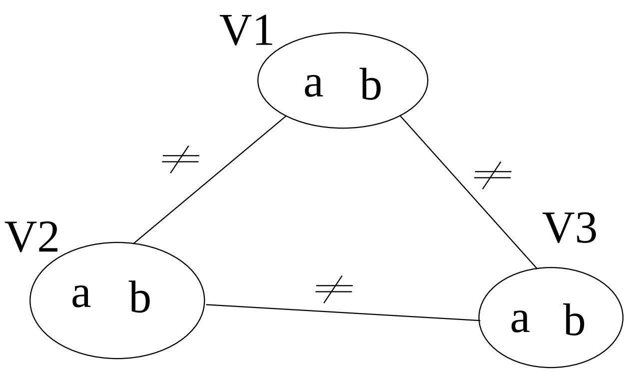
- When a single constraint check is very expensive to make, use AC-7.
- When there is a lot of propagation, use AC-4
- When little propagation and constraint checks are cheap, use AC-3x

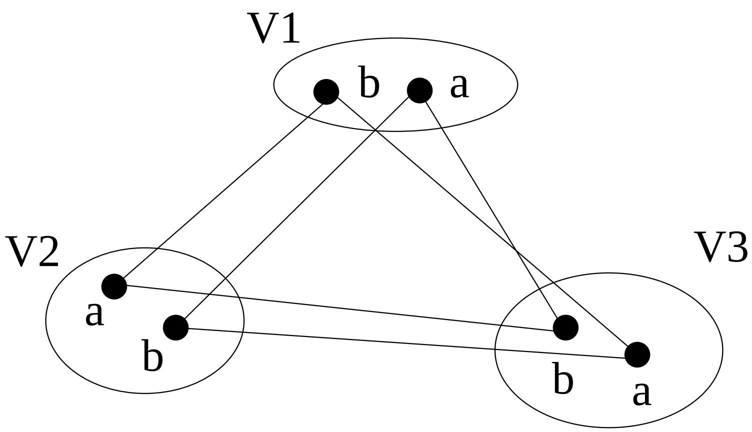
Advice? *Instructor's personal opinion*

Use AC-3..

AC is not enough

Example borrowed from Rina Dechter





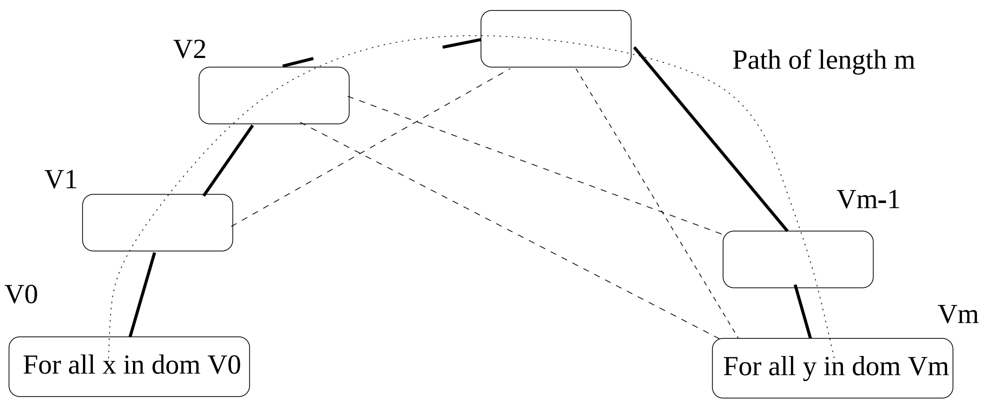
Arc-consistent?

Satisfiable?

$\longrightarrow$ seek higher levels of consistency

Path consistency (I)

A path $(V_0, V_1, V_2, \ldots, V_m)$ of length m is **consistent** iff
for any value x in domain of V_0 and for any value y is domain of Vm, that are consistent (i.e., $P_{V_0 V_m}(x, y)$)
$\exists$ a sequence of values $z_1, z2, \ldots, z_{m-1}$ in the domains of variables $V_1, V_2, \ldots, V_{m-1}$, such that all constraints **among them** are satisfied (i.e., $P_{V_0 V_1}(x, z_1) \wedge P_{V_1 V_2}(z_1, z_2) \wedge \ldots$)



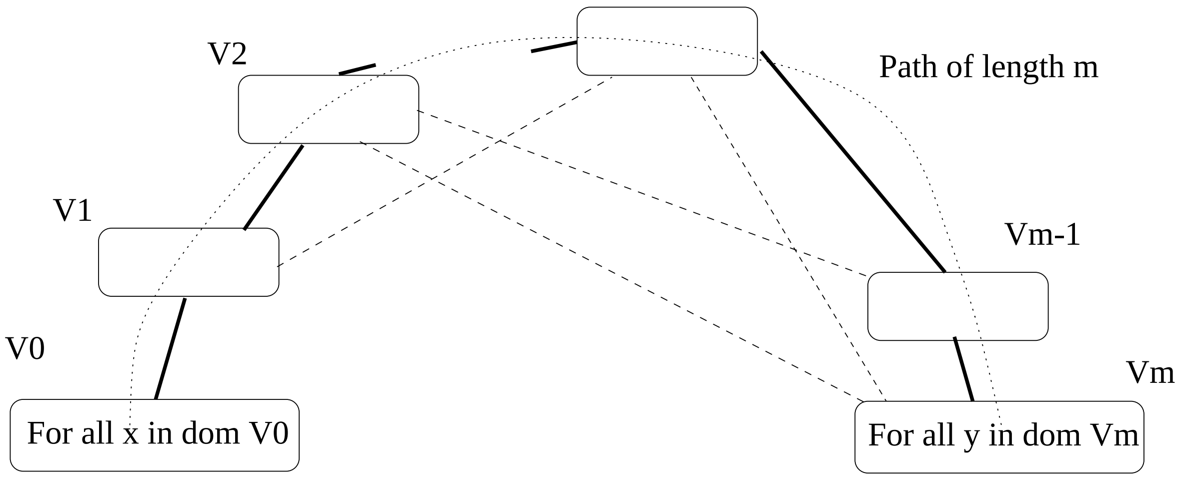
Path consistency (II)

→ The same variable can appear more than once in the path

→ Every time, it may have a different value

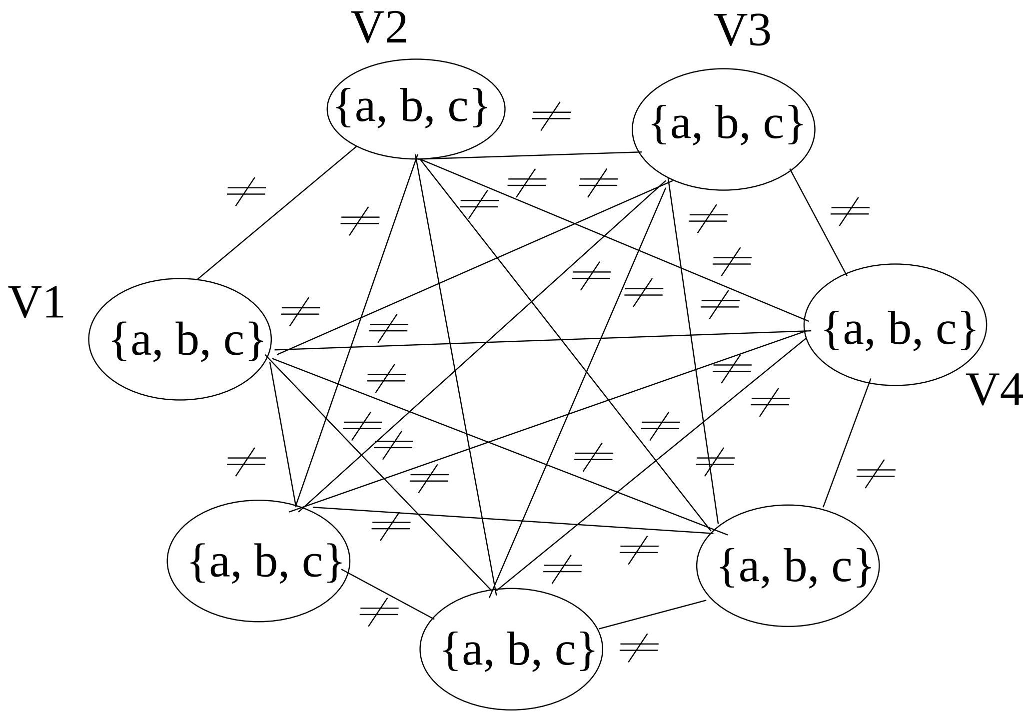
→ Constraints considered: P_{V_0,V_m} and those along the path

→ All other constraints neglected



Path consistency (III)

→ Check path length =2, 3, 4, 5, 6, ...



Definition: A CSP is path consistent iff every path is consistent (i.e., any length of path)

Path consistency (IV)

⟶ A path of length m is path consistent
⟶ A CSP is path consistent

Definition: If every path of any length of a [CSP] is path consistent, then the network is path consistent

Definition: Complete graph

Theorem: In a **complete** graph, if every path of length 2 is consistent, the network is path consistent [Mackworth AIJ'77]
⟶ PC-1
⟶ Proof by induction

Path consistency (PC-1)

Achieved by composition and intersection (of binary relations expressed as matrices) over all paths of length two.

Procedure PC-1:

Begin

  $Y^n \leftarrow R$

  repeat

    begin

    $Y^0 \leftarrow Y^n$

    For $k \leftarrow 1$ **until** n **do**

      For $i \leftarrow 1$ **until** n **do**

        For $j \leftarrow 1$ **until** n **do**

          $Y_{ij}^{k} \leftarrow Y_{ij}^{k-1} \cap Y_{ik}^{k-1} \cdot Y_{kk}^{k-1} \cdot Y_{kj}^{k-1}$

    end

  until $Y^n = Y^0$

  $Y \leftarrow Y^n$

end

Properties of PC-1

Discrete CSPs: [Montanari'74]

1. PC-1 terminates
2. PC-1 results in a path consistent CSP

PC-1 terminates. It is complete, sound (for finding PC network)
PC-2 Improvement similar to (AC-1 $\rightarrow$ AC-3)

Complexity of PC-1..

Complexity of PC-1

Procedure PC-1:

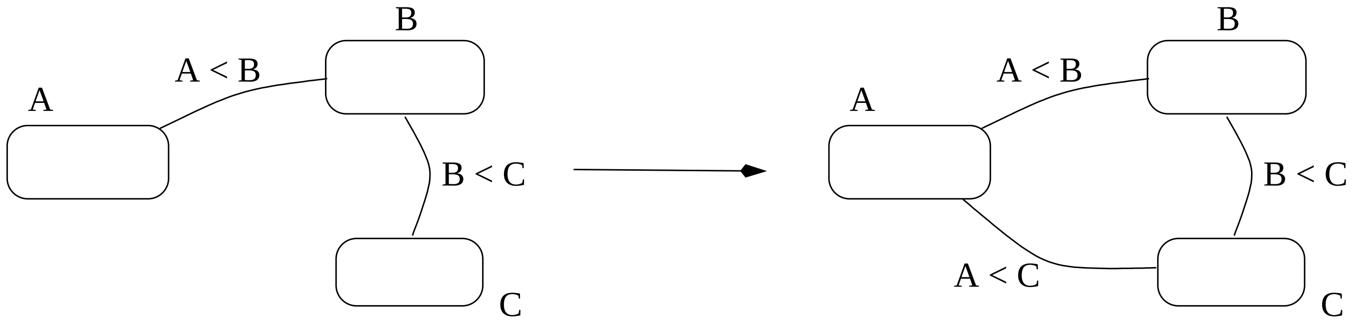
1 **Begin**
2 $Y^n \leftarrow R$
3 **repeat**
4 **begin**
5 $Y^0 \leftarrow Y^n$
6 For $k \leftarrow 1$ **until** n **do**
7 For $i \leftarrow 1$ **until** n **do**
8 For $j \leftarrow 1$ **until** n **do**
9 $Y_{ij}^k \leftarrow Y_{ij}^{k-1} \cap Y_{ik}^{k-1} \cdot Y_{kk}^{k-1} \cdot Y_{kj}^{k-1}$
10 end
11 **until** $Y^n = Y^0$
12 $Y \leftarrow Y^n$
10 **end**

Line 9: a^3
Lines 6–10: $n^3.a^3$
Line 3: at most n^2 relations$\times a^2$ elements

PC-1 is $\mathcal{O}(a^5 n^5)$

PC-3 is $\mathcal{O}(a^5 n^3)$ and $\Omega(a^3 n^3)$

Path consistency as inference of binary constraints



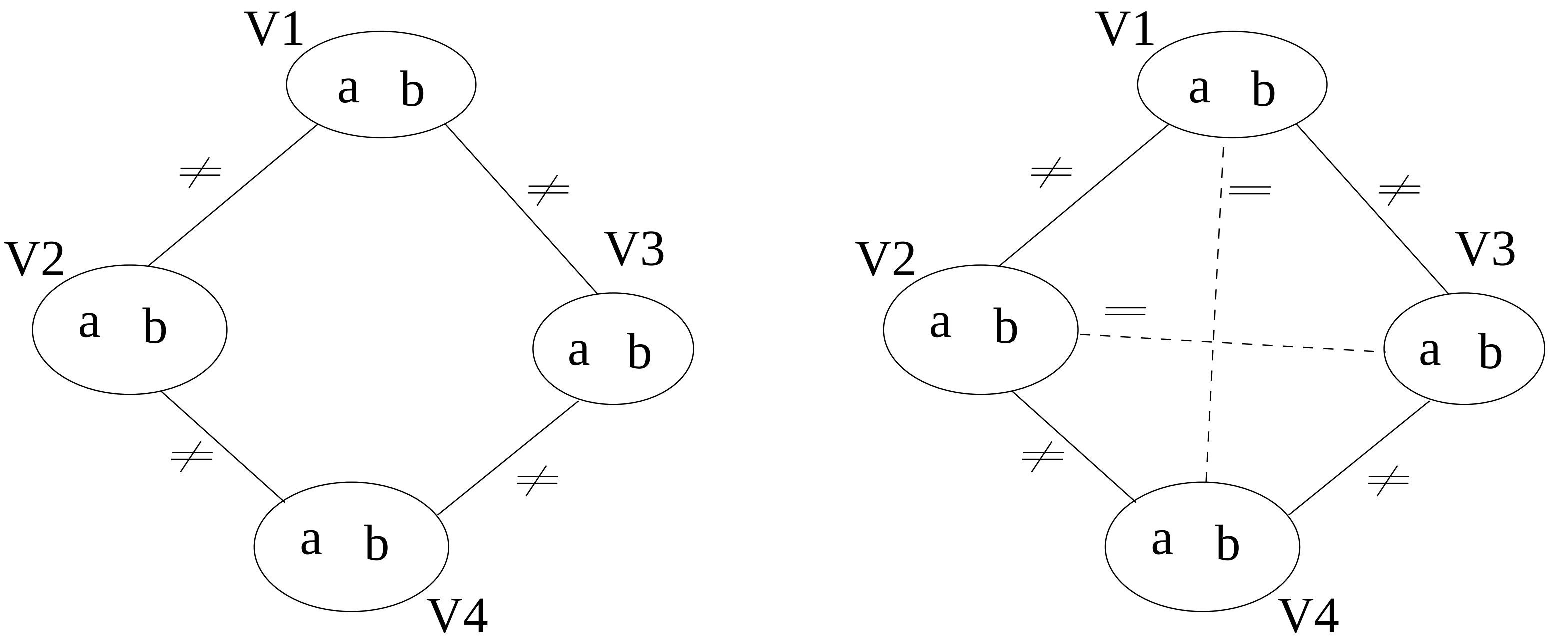
Path-consistency corresponds to inferring a new constraint (alternatively, tightening the existing constraint) between every two variables given the constraints that link them to a third variable

$\longrightarrow$ Considers all subgraphs of 3 variables

$\longrightarrow$ 3-consistency

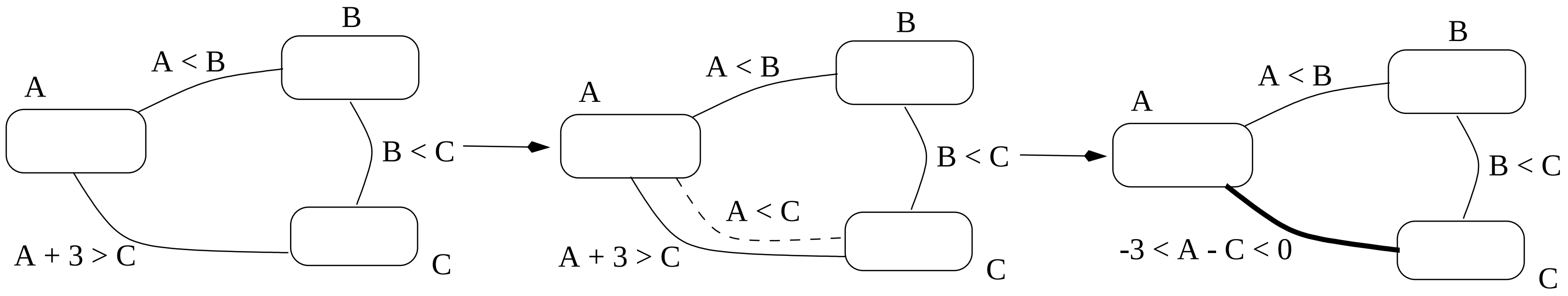
Path consistency as inference of binary constraints

Another example:



Question: *Adapted from Rina Dechter*

Given three variables V_i, V_k, and V_j and the constraints C_{V_i,V_k}, C_{V_i,V_j}, and C_{V_k,V_j}, write the effect of PC as a sequence of operations in relational algebra



Solution:

$$C_{V_i,V_j} \leftarrow C_{V_i,V_j} \cap \pi_{ij}(C_{V_i,V_k} \bowtie C_{V_k,V_j})$$

Path consistency approximates minimality

PC tightens the binary constraints

The tightest possible binary constraints yield the minimal network

Minimal network *a.k.a. central problem*
Given two values for two variables, if they are consistent, then they appear in at least one solution.

Minimal $\Rightarrow$ path consistent

The definition of minimal CSP is concerned with binary CSPs

Minimal CSP

Minimal network *a.k.a. central problem*
Given two values for two variables, if they are consistent, then they appear in at least one solution.

Informally:

In a minimal CSP the remainder of the CSP does not add any further constraint the direct constraint C_{V_i,V_j} between the two variables V_i and V_j [Mackworth AIJ'77]

A minimal CSP is perfectly explicit: as far as the pair V_i and V_j is concerned, the rest of the network does not add any further constraint to the direct constraint C_{V_i,V_j} [Montanari'74]

The binary constraints are explicit as possible. [Montanari'74]

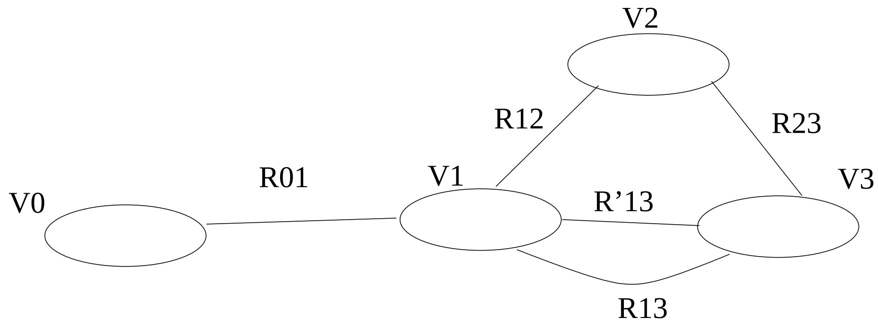
Decomposability generalizes minimality

Any combination of values for k variables that satisfy the constraints between them can be extended to a solution

Decomposable $\Rightarrow$ minimal $\Rightarrow$ path consistent

Path consistency guarantees minimality, decomposability

In PC-1, two operations: $\begin{cases} \text{intersection} & \cap \\ \text{composition} & \cdot \end{cases}$



$$R_{01} \cdot (R_{13} \cap R'_{13}) = (R_{01} \cdot R_{13}) \cap (R_{01} \cdot R'_{13})$$

When composition distributes over intersection then:

[Montanari'74]

1. PC guarantees that CSP is minimal and decomposable
2. 1 loop is sufficient

Temporal reasoning: constraints of bounded difference

Variables: $X, Y, Z, etc.$

Constraints: $a \leq Y - X \leq b$, i.e. $Y - X = [a, b] = I$

Composition: $I_1 \cdot I_2 = [a_1, b_1] \cdot [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$

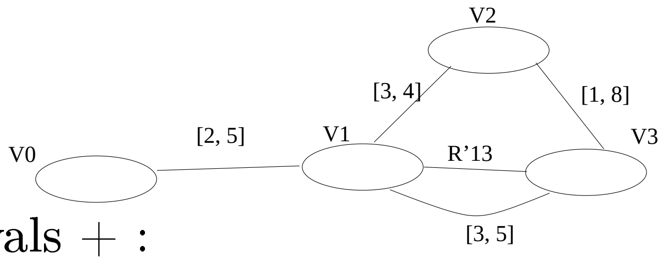
Interpretation: intervals indicate distances, composition is triangle inequality

Intersection: $I_1 \cap I_2 = [max(a_1, a_2), min(b_1, b_2)]$

Distributivity: $I_1 \cdot (I_2 \cap I_3) = (I_1 \cdot I_2) \cap (I_1 \cdot I_3)$

Prove: left as exercise

Example: constraints of bounded difference (temporal reasoning)



Composition of intervals $+$:

$R'_{13} = R_{12} + R_{23} = [4, 12]$

$R_{01} + R_{13} = [2, 5] + [3, 5] = [5, 10]$

$R_{01} + R'_{13} = [2, 5] + [4, 12] = [6, 17]$

Intersection of intervals: $R_{13} \cap R'_{13} = [4, 12] \cap [3, 5] = [4, 5]$

$$R_{01} + (R_{13} \cap R'_{13}) = (R_{01} + R_{13}) \cap (R_{01} + R'_{13})$$

$R_{01} + (R_{13} \cap R'_{13}) = [2, 5] + [4, 5] = [6, 10]$
$(R_{01} + R_{13}) \cap (R_{01} + R'_{13}) = [5, 10] \cap [6, 17] = [6, 10]$

Path consistency guarantees minimality

Side note: PC-1 generalizes Floyd-Warshall algorithm (all-pairs shortest paths), where composition is 'scalar addition' and intersection is 'scalar minimal'

Main improvements

- **Mohr & Henderson** (AIJ 86):
  PC-2 $\mathcal{O}(a^5n^3) \longrightarrow$ PC-3 $\mathcal{O}(a^3n^3)$
  Open question: PC-3 optimal?
- **Han & Lee** (AIJ 88):
  PC-3 is incorrect
  PC-4 $\mathcal{O}(a^3n^3)$ space and time

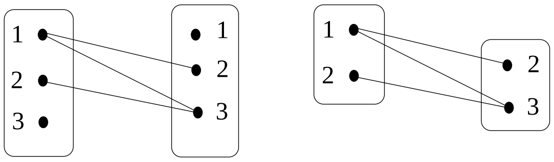
In practical applications:

PC is seldom used in practical applications, unless in presence of special type of constraints (e.g., bounded difference)

Constraint propagation

(courtesy of R. Dechter, UC-Irvine)

After Arc-consistency:

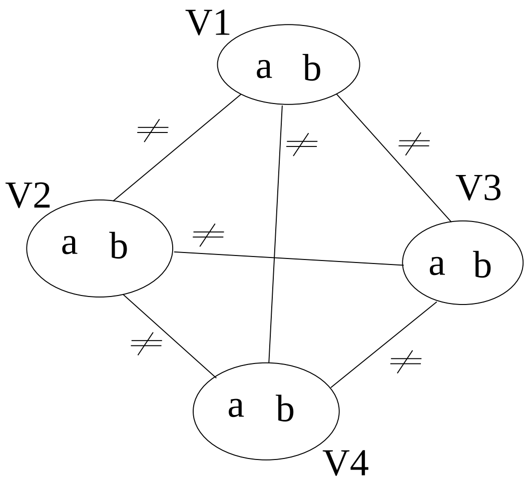


After Path-consistency:

Are these CSPs the same?
Which one is more explicit?
Are they equivalent?

The more propagation,
the more explicit the constraints,
the more search is directed towards a solution.

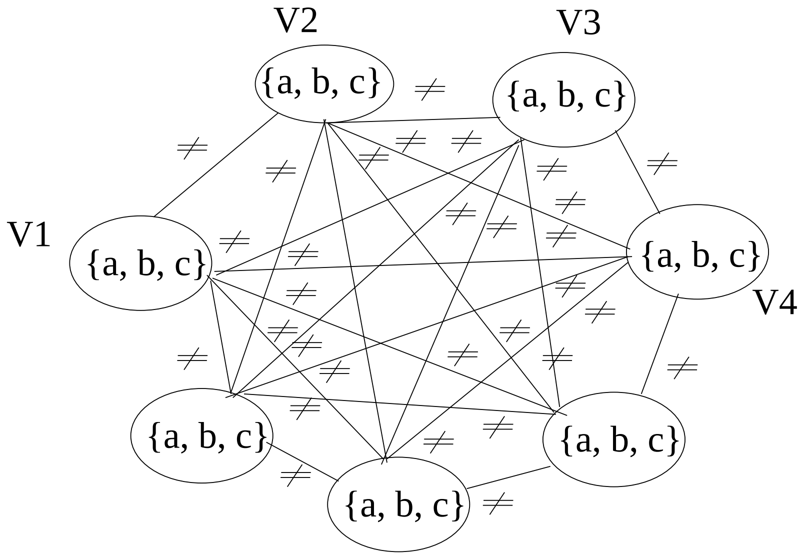
PC can detect unsatisfiability



Arc-consistent?

Path-consistent?

PC is not enough



Arc-consistent?

Path-consistent?

Satisfiable?

$\longrightarrow$ seek (even) higher levels of consistency

$\longrightarrow$ k-consistency, $k = 1, 2, 3, \ldots$

k-consistency

- 2-consistency: $D_{V_i} \leftarrow D_{V_i} \cap \pi_{V_i}(C_{V_i,V_j} \bowtie D_{V_j})$
  Projection of binary constraint is used to tighten unary constraint
- 3-consistency: $C_{V_i,V_j} \leftarrow C_{V_i,V_j} \cap \pi_{V_i,V_j}(C_{V_i,V_k} \bowtie C_{V_k,V_j})$
  Projection of ternary constraint is used to tighten binary constraint
- k-consistency: [Freuder CACM'78, JACM'82]
  Projection of k-ary constraint is used to tighten $(k-1)$-ary constraint

k-**consistency**: definition

A CSP is k-consistent iff given any consistent instantiation of any $(k-1)$ distinct variables, there exists an instantiation of any k^{th} variable such that the k values taken together satisfy all the constraints among the k variables.

Example: *(courtesy of R. Dechter, UC-Irvine)*

3-consistent?

Q			
	Q		

4-consistent?

Q			
		Q	
	Q		

Summary: the higher the consistency level, the better

- New terms: Waltz algorithm (Refine, Revise), AC (Revise), PC, path consistent, complete graph, minimal CSP, decomposable CSP
- Complexity of AC, PC algorithms
- AC tightens domains, PC tightens binary constrains
- Consistency operations in terms of composition, intersection and relational algebra
- Path consistency approximates minimality, which approximates decomposability
- One situation when path consistency guarantees minimality and decomposability
- Criteria for performance comparison
- Parameters for random CSPs

Related results

- Certain graph topologies guarantee minimality
- Higher consistence levels: (strong) k-consistency, (i, j)-consistency
- Non-binary constraints: GAC
- Integration with search: FC, MAC
- Continuous domains: box consistency
- etc.