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Graph Minors.
X. Obstructions to Tree-Decomposition

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Roughly, a graph has small "tree-width" if it can be constructed by piecing small graphs together in a tree structure. Here we study the obstructions to the existence of such a tree structure. We find, for instance:

(i) a minimax formula relating tree-width with the largest such obstructions

(ii) an association between such obstructions and large grid minors of the graph

(iii) a "tree-decomposition" of the graph into pieces corresponding with the obstructions.

These results will be of use in later papers.

1. TANGLES

Graphs in this paper are finite and undirected and may have loops or multiple edges. The vertex- and edge-sets of a graph G are denoted by $V(G)$ and $E(G)$. If $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are subgraphs of a graph G, we denote the graphs $(V_1 \cap V_2, E_1 \cap E_2)$ and $(V_1 \cup V_2, E_1 \cup E_2)$ by $G_1 \cap G_2$ and $G_1 \cup G_2$, respectively. A *separation* of a graph G is a pair (G_1, G_2) of subgraphs with $G_1 \cup G_2 = G$ and $E(G_1 \cap G_2) = \emptyset$, and the *order* of this separation is $|V(G_1 \cap G_2)|$.

It sometimes happens with a graph G that for each separation (G_1, G_2) of G of low order, we may view one of G_1, G_2 as the "main part" of G, in

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a consistent way. For example if G is drawn on a connected surface (not a sphere) and every non-null-homotopic curve in the surface meets the drawing many times, then it can be shown (see [5]) that for each low order separation (G_1, G_2), exactly one of G_1, G_2 contains a non-null-homotopic circuit. As a second example, let H be a minor of G (defined later), isomorphic to a large complete graph; then for each low order separation (G_1, G_2) of G, exactly one of G_1, G_2 has a subgraph corresponding to a vertex of H. The object of this paper is to study such "tangles," as we call them, since they play a central role in future papers of this series.

Many of our results about tangles extend easily to hypergraphs, and we have expressed them in this generality. A *hypergraph* G consists of a set of *vertices* $V(G)$, a set of *edges* $E(G)$, and an incidence relation; each edge may or may not be incident with each vertex. If each edge is incident with either one or two vertices, the hypergraph is a *graph*. All hypergraphs in this paper are finite. A *subhypergraph* G' of G is a hypergraph such that

(i) $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$

(ii) for $e \in E(G')$ and $v \in V(G)$, e is incident with v in G if and only if $v \in V(G')$ and e is incident with v in G'.

We write $G' \subseteq G$ if G' is a subhypergraph of G. We define $G_1 \cup G_2$, $G_1 \cap G_2$ for subhypergraphs G_1, G_2 of a hypergraph as for graphs, and a separation of a hypergraph, and its order, are defined as for graphs. If G is a hypergraph and $X \subseteq E(G)$, $G \backslash X$ is the subhypergraph G' with $V(G') = V(G)$, $E(G') = E(G) - X$; while if $X \subseteq V(G)$, $G \backslash X$ is the subhypergraph with $V(G') = V(G) - X$ and $E(G')$ the set of those edges of G incident with no vertex in X. We sometimes abbreviate $G \backslash \{x\}$ to $G \backslash x$, etc.

Let G be a hypergraph and let $\theta \geqslant 1$ be an integer. A *tangle* in G of *order* θ is a set $\mathscr{T}$ of separations of G, each of order $< \theta$, such that

(i) for every separation (A, B) of G of order $< \theta$, one of (A, B), (B, A) is in $\mathscr{T}$

(ii) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathscr{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$

(iii) if $(A, B) \in \mathscr{T}$ then $V(A) \neq V(G)$.

We refer to these as the *first*, *second*, and *third* (*tangle*) *axioms*. Every tangle $\mathscr{T}$ has order $\leqslant |V(G)|$, since $(G, G \backslash E(G))$, $(G \backslash E(G), G) \notin \mathscr{T}$. The *tangle number* of G, denoted $\theta(G)$, is the maximum order of tangles in G (or 0, if there are no tangles).

The main results of this paper are as follows:

(1) Tangle number is connected with "tree-width," which was discussed in earlier papers of this series (for example, [3]); indeed, there is a

minimax equation connecting the tangle number of a hypergraph and its "branch-width," which is an invariant very similar to tree-width and essentially within a constant factor of tree-width.

(2) Despite our rather abstract definition of a tangle, there are in any hypergraph G at most $|V(G)|$ maximal tangles, and any other tangle is a subset (a "truncation") of one of these. Furthermore, there is a "tree-decomposition" of G, the vertices of which correspond to these maximal tangles.

(3) For $\theta \geqslant 2$, any minor isomorphic to a $(\theta \times \theta)$-grid of a graph G gives rise to a tangle in G of order θ, and conversely, for any $\theta \geqslant 2$ there exists $N(\theta) \geqslant \theta$ such that for every tangle of order $\geqslant N(\theta)$ in a graph G, its truncation to order θ is the tangle arising from some $(\theta \times \theta)$-grid minor of G.

(4) Finally, the main result of the paper. It is too technical to state without a number of definitions, but roughly it enables us to gain knowledge of the global structure of a hypergraph from a knowledge of its structure relative to each tangle. This will be applied in [6].

2. Some Tangle Lemmas

In this section we develop some easy results about tangles for later use.

(2.1) *If $\mathscr{T}$ is a tangle and $(A, B) \in \mathscr{T}$ then $(B, A) \notin \mathscr{T}$.*

Proof. Since $A \cup B = G$, $(B, A) \notin \mathscr{T}$ by the second axiom. ∎

(2.2) *If $\mathscr{T}$ is a tangle of order θ and (A, B), $(A', B') \in \mathscr{T}$ and $(A \cup A', B \cap B')$ has order $< \theta$ then $(A \cup A', B \cap B') \in \mathscr{T}$.*

Proof. Now $(B \cap B', A \cup A') \notin \mathscr{T}$ by the second axiom, because (A, B), $(A', B') \in \mathscr{T}$ and $A \cup A' \cup (B \cap B') = G$. Thus $(A \cup A', B \cap B') \in \mathscr{T}$ by the first axiom. ∎

(2.3) *If $\mathscr{T}$ has order $\geqslant 2$ and (A_1, B_1), (A_2, B_2), $(A_3, B_3) \in \mathscr{T}$ then $E(A_1 \cup A_2 \cup A_3) \neq E(G)$.*

Proof. Suppose that there exist (A_1, B_1), (A_2, B_2), $(A_3, B_3) \in \mathscr{T}$ with $E(A_1 \cup A_2 \cup A_3) = E(G)$, and choose them with $|V(A_1)|$ maximum. By the second axiom, $A_1 \cup A_2 \cup A_3 \neq G$, and so there is a vertex v of G in none of $V(A_1)$, $V(A_2)$, $V(A_3)$ and hence incident with no edge of G. Let K be the hypergraph with $V(K) = \{v\}$, $E(K) = \varnothing$. Then (K, G) has order 1 and by the second axiom, $(G, K) \notin \mathscr{T}$; thus $(K, G) \in \mathscr{T}$ by the first axiom, since $\mathscr{T}$

has order ≥ 2. Now $(K, G\backslash v)$ has order 0, and $(G\backslash v, K) \notin \mathscr{T}$ by the second axiom, since $(G\backslash v) \cup K = G$. Thus $(K, G\backslash v) \in \mathscr{T}$. But $(K \cup A_1, (G\backslash v) \cap B_1)$ has order at most the order of (A_1, B_1) and hence is in $\mathscr{T}$ by (2.2), contrary to the maximality of $|V(A_1)|$, as required. ▮

For an edge e of a hypergraph G, the *ends* of e are the vertices of G incident with e, and the *size* of e is the number of ends of e.

(2.4) *Let $\theta \geq 1$, and let e be an edge of G with size $\geq \theta$. Let $\mathscr{T}$ be the set of all separations (A, B) of G of order $< \theta$ with $e \in E(B)$. Then $\mathscr{T}$ is a tangle of order θ.*

Proof. The first two axioms are clear. For the third, let $(A, B) \in \mathscr{T}$. Then $V(A \cap B)$ does not contain every end of e since $|V(A \cap B)| < \theta$, and yet $e \in E(B)$, and so $V(A) \neq V(G)$. This completes the proof. ▮

We remark

(2.5) *G has a tangle if and only if $V(G) \neq \varnothing$.*

Proof. If $v \in V(G)$, let $\mathscr{T}$ be the set of all separations (A, B) of G of order 0 with $v \in V(B)$. Then $\mathscr{T}$ is a tangle of order 1, as is easily seen. Conversely, since every tangle has order $\leq |V(G)|$, if G has a tangle then $V(G) \neq \varnothing$. ▮

For graphs, we can extend (2.5) as follows.

(2.6) *If G is a graph, the tangles in G of order 1 are in 1–1 correspondence with the connected components of G, and those of order 2 are in 1–1 correspondence with the blocks of G which have a non-loop edge.*

(A *block* of a graph is a maximal connected subgraph any two distinct edges of which are in a circuit.)

Proof. Since we do not need the result, we merely sketch the proof. Any $v \in V(G)$ yields a tangle of order 1 as in (2.5), and it is easy to see that every tangle of order 1 arises this way, and distinct $v, v' \in V(G)$ yield the same tangle if and only if v and v' are in the same component of G. For order 2, any non-loop edge yields a tangle of order 2, by (2.4), and again, it is easy to see that every order 2 tangle arises this way, and two edges yield the same tangle if and only if they are in the same block. ▮

One might speculate that in a graph, the tangles of order d correspond to the long-sought "d-connected components," but that possibility is not further explored here.

Some further lemmas:

(2.7) *Let $\mathscr{T}$ be a set of separations of a hypergraph G, each of order $<\theta$, satisfying the first and second tangle axioms. Then $\mathscr{T}$ is a tangle if and only if $(K_e, G\backslash e) \in \mathscr{T}$ for every $e \in E(G)$ of size $<\theta$, where K_e is the hypergraph formed by e and its ends.*

Proof. If $\mathscr{T}$ is a tangle and $e \in E(G)$ then $(G\backslash e, K_e) \notin \mathscr{T}$ by the third tangle axiom, since $V(G\backslash e) = V(G)$, and so $(K_e, G\backslash e) \in \mathscr{T}$, as required. For the converse, let $\mathscr{T}$ not be a tangle, and choose $(A, B) \in \mathscr{T}$ with $V(A) = V(G)$ and with B minimal. By the second tangle axiom, $A \neq G$ and so $E(B) \neq \varnothing$; choose $e \in E(B)$. From the minimality of B, $(A \cup K_e, B\backslash e) \notin \mathscr{T}$, and so $(B\backslash e, A \cup K_e) \in \mathscr{T}$. Hence $(K_e, G\backslash e) \notin \mathscr{T}$ by the second axiom, since $(A, B) \in \mathscr{T}$ and $A \cup (B\backslash e) \cup K_e = G$. But e has size $<\theta$, since every end of e is in $V(A \cap B)$. The result follows. ▮

Let $\mathscr{T}$ be a tangle in a hypergraph G. A separation $(A, B) \in \mathscr{T}$ is *extreme* if $A' = A$ and $B' = B$ for every $(A', B') \in \mathscr{T}$ with $A \subseteq A'$ and $B' \subseteq B$.

(2.8) *Let $\mathscr{T}$ be a tangle of order θ in a hypergraph G, and let $(A, B) \in \mathscr{T}$ be extreme. Then (A, B) has order $\theta - 1$. Moreover, if (B_1, B_2) is a separation of B, then either $B_1 \subseteq A \cap B$ and $B_2 = B$, or $B_2 \subseteq A \cap B$ and $B_1 = B$, or (B_1, B_2) has order strictly greater than*

$$\min(|V(A \cap B_1)|, |V(A \cap B_2)|).$$

In particular, there is no separation (B_1, B_2) of B with B_1, B_2 non-null of order 0, and there is no edge of B with all its ends in $V(A)$.

Proof. By the third axiom there exists $v \in V(B) - V(A)$. Let K_v be the hypergraph with vertex set $\{v\}$ and with no edges. From the extremity of (A, B), $(A \cup K_v, B) \notin \mathscr{T}$, and $(B, A \cup K_v) \notin \mathscr{T}$ by the second axiom, since $(A, B) \in \mathscr{T}$ and $A \cup B = G$. Thus $(A \cup K_v, B)$ has order $\geqslant \theta$, and so (A, B) has order $\theta - 1$.

Let (B_1, B_2) be a separation of B. If $(A \cup B_1, B_2) = (A, B)$ then $B_1 \subseteq A$ and $B_2 = B$, and so we may assume that $(A \cup B_1, B_2) \neq (A, B)$. From the extremity of (A, B), $(A \cup B_1, B_2) \notin \mathscr{T}$, and similarly $(A \cup B_2, B_1) \notin \mathscr{T}$. Not both $(B_2, A \cup B_1)$, $(B_1, A \cup B_2) \in \mathscr{T}$, by the second axiom, since $A \cup B_1 \cup B_2 = G$, and without loss of generality we assume that $(B_2, A \cup B_1) \notin \mathscr{T}$. Since $(A \cup B_1, B_2) \notin \mathscr{T}$ it follows that $(A \cup B_1, B_2)$ has order $\geqslant \theta$; that is,

$$|V(B_1 \cap B_2)| + |V(A \cap B) - V(A \cap B_1)| \geqslant \theta = |V(A \cap B)| + 1.$$

Hence $|V(B_1 \cap B_2)| > |V(A \cap B_1)|$, as required.

It follows that there is no separation (B_1, B_2) of B of order 0 with B_1, B_2 non-null. Suppose that $e \in E(B)$ has all its ends in $V(A)$. Let K_e be the hypergraph with edge set $\{e\}$ and vertex set the set of ends of e; then $(K_e, B\backslash e)$ is a separation of B. Now $K_e \not\subseteq A$ since $e \notin E(A)$, and $B\backslash e \not\subseteq A$ since $V(A) \neq V(G)$, and so

$$|V(K_e \cap (B\backslash e))| > \min(|V(A \cap K_e)|, |V(A \cap (B\backslash e))|).$$

But the left side is the number of ends of e, and so is the right side, a contradiction. Thus there is no such e. ∎

(2.9) *Let $\mathscr{T}$ be a tangle of order θ in a hypergraph G, and let $(A_1, B_1) \in \mathscr{T}$. Let (A_2, B_2) be a separation of order $< \theta$. If either*

(i) $V(B_1) \subseteq V(B_2)$, *or*

(ii) $V(A_2) \subseteq V(A_1)$, *or*

(iii) $\theta \geqslant 2$ *and* $E(A_2) \subseteq E(A_1)$ (*equivalently*, $E(B_1) \subseteq E(B_2)$)

then $(A_2, B_2) \in \mathscr{T}$.

Proof. Suppose not; then $(B_2, A_2) \in \mathscr{T}$. Choose $(A, B) \in \mathscr{T}$, extreme, with $B_2 \subseteq A$ and $B \subseteq A_2$. Then $A \cup A_1 \neq G$ by the second axiom. Since $A \cup B = G$ and $A_1 \cup B_1 = G$ it follows that $B \not\subseteq A_1$ and $B_1 \not\subseteq A$.

Case 1. $V(B_1) \subseteq V(B_2)$.

Then $V(B_1) \subseteq V(B_2) \subseteq V(A)$, and $E(B_1) \cap E(B) = \varnothing$, since from (2.8) every edge of B has an end in $V(G) - V(A) \subseteq V(G) - V(B_1)$. Thus $E(B_1) \subseteq E(A)$ and so $B_1 \subseteq A$, a contradiction.

Case 2. $V(A_2) \subseteq V(A_1)$.

Since $(B_2, A_2) \in \mathscr{T}$ and (B_1, A_1) has order $< \theta$, and $V(A_2) \subseteq V(A_1)$, it follows that $(B_1, A_1) \in \mathscr{T}$, since the theorem holds in Case 1. But this contradicts (2.1).

Case 3. $\theta \geqslant 2$ and $E(A_2) \subseteq E(A_1)$.

Since $E(B) \subseteq E(A_2) \subseteq E(A_1)$ and $B \not\subseteq A_1$, there is a vertex v of B with $v \notin V(A_1)$. Since $E(B) \subseteq E(A_1)$, it follows that v is incident with no edge of B. By (2.8), $V(B) = \{v\}$ and $E(B) = \varnothing$, and since $V(A) \neq V(G)$, it follows that $V(A \cap B) = \varnothing$. By (2.8) again, $\theta = 1$, a contradiction. ∎

For future reference, we observe the following.

(2.10) *Let $\mathscr{T}$ be a tangle of order $\geqslant 3$ in a graph G, and let $(A, B) \in \mathscr{T}$. Then B has a circuit.*

Proof. It suffices to prove the result when (A, B) is extreme. By (2.8), $|A \cap B| \geqslant 2$; let $v_1, v_2 \in V(A \cap B)$ be distinct.

(1) *There is no separation* (B_1, B_2) *of* B *of order* $\leqslant 1$ *with* $v_1 \in V(B_1) - V(B_2)$ *and* $v_2 \in V(B_2) - V(B_1)$.

For such a separation would satisfy

$$\min(|V(A \cap B_1)|, |V(A \cap B_2)|) \geqslant 1$$

and $B_1, B_2 \neq B$, contrary to (2.8).

Moreover, from (2.8), v_1 and v_2 are not adjacent in B. From (1) and Menger's theorem, there are two paths of B between v_1 and v_2, internally disjoint, and hence B has a circuit, as required. ▌

3. A Lemma about Submodular Functions

Now we turn to our first main result, the minimax theorem relating tangle number and branch-width. It is most convenient to prove a generalization, which is a statement about submodular functions.

Let E be a finite set. A *connectivity function on* E is a function κ from the set of all subsets of E to the set of integers such that

(i) for $X \subseteq E$, $\kappa(X) = \kappa(E - X)$

(ii) for $X, Y \subseteq E$, $\kappa(X \cup Y) + \kappa(X \cap Y) \leqslant \kappa(X) + \kappa(Y)$.

For instance, if G is a hypergraph and $E = E(G)$, we would let $\kappa(X)$ be the number of vertices of G incident both with an edge in X and with an edge in $E - X$; or if M is a matroid with rank function r and $E = E(M)$, we could let $\kappa(X) = r(X) + r(E - X)$.

A subset $X \subseteq E$ is *efficient* if $\kappa(X) \leqslant 0$. A *bias* is a set $\mathscr{B}$ of efficient sets, such that

(i) if $X \subseteq E$ is efficient then $\mathscr{B}$ contains one of $X, E - X$

(ii) if $X, Y, Z \in \mathscr{B}$ then $X \cup Y \cup Z \neq E$.

A bias $\mathscr{B}$ is said to *extend* a set $\mathscr{A}$ of efficient sets if $\mathscr{A} \subseteq \mathscr{B}$. We are concerned with the problem of, given $\mathscr{A}$, when is there a bias extending $\mathscr{A}$?

Let us describe an obstacle to the existence of such a bias. A *tree* is a connected non-null graph with no circuits; its vertices of valency $\leqslant 1$ are its *leaves*. A tree is *ternary* if every vertex has valency 1 or 3. (Thus, ternary trees have $\geqslant 2$ leaves.) An *incidence* in a tree T is a pair (v, e), where $v \in V(T)$, $e \in E(T)$, and e is incident with v. A *tree-labelling over* $\mathscr{A}$ is a pair (T, α), where T is a ternary tree, and α is a function from the set of all incidences in T to the set of efficient subsets of E, such that

(i) for each $e \in E(T)$ with ends u, v, say, $\alpha(u, e) = E - \alpha(v, e)$

(ii) for each incidence (v, e) in T such that v is a leaf, either $\alpha(v, e) = E$ or $\alpha(v, e) \cup X = E$ for some $X \in \mathscr{A}$

(iii) if $v \in V(T)$ has valency 3, incident with e_1, e_2, e_3, say, then $\alpha(v, e_1) \cup \alpha(v, e_2) \cup \alpha(v, e_3) = E$.

(3.1) *If there is a bias extending $\mathscr{A}$ then there is no tree-labelling over $\mathscr{A}$.*

Proof. Suppose that $\mathscr{B}$ is a bias extending $\mathscr{A}$, and (T, α) is a tree-labelling over $\mathscr{A}$. An incidence (v, e) of T is *passive* if $\alpha(v, e) \notin \mathscr{B}$. For each edge e with ends u, v, $\mathscr{B}$ contains exactly one of $\alpha(u, e)$, $\alpha(v, e)$ since they are efficient complementary sets. Thus there are precisely $|E(T)|$ passive incidences. Since T has $|E(T)| + 1$ vertices there is a vertex v of T in no passive incidence; that is, $\alpha(v, e) \in \mathscr{B}$ for all edges e incident with v. If v has valency 1 then by the definition of a tree-labelling, either $\alpha(v, e) = E$ or $\alpha(v, e) \cup X = E$ for some $X \in \mathscr{A}$, in either case contrary to the definition of a bias. Thus v has valency 3. Let e_1, e_2, e_3 be the edges of T incident with v; then

$$\alpha(v, e_1) \cup \alpha(v, \alpha_2) \cup \alpha(v, e_3) = E$$

by the definition of a tree-labelling, and yet each $\alpha(v, e_i) \in \mathscr{B}$, contrary to the definition of a bias, as required. ∎

The main result of this section is a converse of (3.1), in a strong form, that if there is no bias extending $\mathscr{A}$, then there is an exact tree-labelling over $\mathscr{A}$. "Exact" is defined as follows. Let (T, α) be a tree-labelling over $\mathscr{A}$. A *fork* in T is an unordered pair $\{e_1, e_2\}$ of distinct edges of T with a common end (the *nub* of the fork). A fork $\{e_1, e_2\}$ with nub t is *exact* (for α) if $\alpha(t, e_1) \cap \alpha(t, e_2) = \emptyset$. We say that (T, α) is *exact* if every fork of T is exact. We require the following lemma.

(3.2) *If there is a tree-labelling over $\mathscr{A}$ then there is an exact tree-labelling over $\mathscr{A}$, using the same tree.*

Proof. Choose a tree T such that there is a tree-labelling (T, α) over $\mathscr{A}$. Choose $t_0 \in V(T)$. For each $t \in V(T)$ we denote by $d(t)$ the number of edges in the path of T between t_0 and t. Choose α satisfying (1), (2), and (3), below.

(1) *(T, α) is a tree-labelling over $\mathscr{A}$.*

(2) *Subject to* (1), $\sum \kappa(\alpha(v, e))$ (*summed over all incidences (v, e) of T*) *is minimum.*

(3) *Subject to* (1) *and* (2), $\sum 3^{-d(t)}$ (*summed over all non-exact forks, where t is the nub of the fork*) *is minimum.*

We claim that (T, α) is exact. For suppose that some fork $\{e_1, e_2\}$ with nub t is non-exact. Then t has valency 3 in T, since T is ternary; let e_3 be the third edge of T incident with v, and let e_i have ends t, t_i $(i = 1, 2, 3)$. Let $A_1 = \alpha(t, e_1)$, $A_2 = \alpha(t, e_2)$. Define α' by

$$\alpha'(t, e_1) = A_1 - A_2$$

$$\alpha'(t_1, e_1) = \alpha(t_1, e_1) \cup A_2 = E - (A_1 - A_2)$$

$$\alpha'(v, e) = \alpha(v, e) \text{ for } (v, e) \neq (t, e_1), (t_1, e_1).$$

We claim that $\kappa(A_1 - A_2) \geqslant \kappa(A_1)$. For if $\kappa(A_1 - A_2) \geqslant 0$ this is true, and so we may assume that $A_1 - A_2$ is efficient. Then α' is a tree-labelling over $\mathscr{A}$, and from (2),

$$\kappa(\alpha'(t, e_1)) + \kappa(\alpha'(t_1, e_1)) \geqslant \kappa(\alpha(t, e_1)) + \kappa(\alpha(t_1, e_1));$$

that is,

$$\kappa(A_1 - A_2) + \kappa(E - (A_1 - A_2)) \geqslant \kappa(A_1) + \kappa(E - A_1).$$

Since $\kappa(E - (A_1 - A_2)) = \kappa(A_1 - A_2)$ and $\kappa(E - A_1) = \kappa(A_1)$, it follows that $\kappa(A_1 - A_2) \geqslant \kappa(A_1)$, as claimed. Similarly $\kappa(A_2 - A_1) \geqslant \kappa(A_2)$. But since κ is a connectivity function,

$$\kappa(A_1) + \kappa(E - A_2) \geqslant \kappa(A_1 \cup (E - A_2)) + \kappa(A_1 \cap (E - A_2));$$

that is,

$$\kappa(A_1) + \kappa(A_2) \geqslant \kappa(A_2 - A_1) + \kappa(A_1 - A_2).$$

Thus equality holds throughout, and in particular, $\kappa(A_1 - A_2) = \kappa(A_1)$ and $\kappa(A_2 - A_1) = \kappa(A_2)$. From the symmetry between t_1 and t_2, we may assume that $d(t) < d(t_1)$. With α' as before we see that α' is a tree-labelling over $\mathscr{A}$ and $\sum \kappa(\alpha'(v, e)) = \Sigma\, \kappa(\alpha(v, e))$. Moreover, $\{e_1, e_2\}$ is exact for α', and any fork of T which is exact for α is exact for α' except possibly for forks $\{e, e_1\}$ with nub t_1. There are at most two such forks, and since $d(t_1) > d(t)$, this contradicts (3), as required. ∎

(3.3) *Let (T, α) be an exact tree-labelling over $\mathscr{A}$, and let (u, f) be an incidence in T. Let T_0 be the component of $T \backslash f$ which contains u. Then, as (v, e) ranges over all incidences of T such that v is a leaf of T and $v \in V(T_0)$, the sets $E - \alpha(v, e)$ are mutually disjoint and have union $E - \alpha(u, f)$.*

Proof. We proceed by induction on $|V(T_0)|$. If u is a leaf the result is trivial, and so we may assume that u is incident with three edges f, f_1, f_2; let f_i have ends u, u_i $(i = 1, 2)$, and let T_i be the component of

$T\backslash f_i$ containing u_i ($i = 1, 2$). Then $V(T_0) = V(T_1) \cup V(T_2) \cup \{u\}$ and $V(T_1) \cap V(T_2) = \varnothing$. Now the result holds for (u_1, f_1) and (u_2, f_2) by the inductive hypothesis. Moreover, since $E - \alpha(u_i, f_i) = \alpha(u, f_i)$ ($i = 1, 2$) and (T, α) is exact, it follows that

$$(E - \alpha(u_1, f_1)) \cup (E - \alpha(u_2, f_2)) = E - \alpha(u, f)$$

$$(E - \alpha(u_1, f_1)) \cap (E - \alpha(u_2, f_2)) = \varnothing.$$

The result follows. ∎

(3.4) *If there is no bias extending $\mathscr{A}$ then there is an exact tree-labelling over $\mathscr{A}$.*

Proof. By (3.2), it suffices to prove that there is a tree-labelling over $\mathscr{A}$. Suppose that $E = \varnothing$. If $\varnothing$ is efficient, let T be a two-vertex tree, and let $\alpha(v, e) = \varnothing$ for both incidences (v, e) of T; (T, α) is the required tree-labelling. If $\varnothing$ is not efficient, then $\mathscr{A}$ is a bias, a contradiction. Thus we may assume that $E \neq \varnothing$. Choose $x \in E$, and let $\mathscr{B}$ be the set of all efficient sets $B \subseteq E$ with $x \notin B$; then $\mathscr{B}$ is a bias. Since $\mathscr{B}$ does not extend $\mathscr{A}$, it follows that $\mathscr{A} \neq \varnothing$.

We proceed by induction on the number N of efficient sets $X \subseteq E$ such that neither X nor $E - X$ is a subset of any member of $\mathscr{A}$. We suppose first that $N = 0$. Let $\mathscr{B}$ be the set of all efficient sets which are subsets of members of $\mathscr{A}$. Since $\mathscr{A} \subseteq \mathscr{B}$, $\mathscr{B}$ is not a bias. But for every efficient set X, either $X \in \mathscr{B}$ or $E - X \in \mathscr{B}$ since $N = 0$. Thus there exist $X_1, X_2, X_3 \in \mathscr{B}$ with $X_1 \cup X_2 \cup X_3 = E$. Let T be the tree with four vertices t_0, t_1, t_2, t_3 and edges e_i with ends t_0, t_i ($i = 1, 2, 3$). Define $\alpha(t_0, e_i) = X_i$, $\alpha(t_i, e_i) = E - X_i$ ($i = 1, 2, 3$). Then (T, α) is a tree-labelling over $\mathscr{A}$, as required.

Thus we may assume $N > 0$. Choose an efficient set $X \subseteq E$ such that neither X nor $E - X$ is a subset of any member of $\mathscr{A}$, and subject to that with X minimal. Since $\mathscr{A} \neq \varnothing$, $X \neq \varnothing$. Let $\mathscr{A}_1 = \mathscr{A} \cup \{X\}$, $\mathscr{A}_2 = \mathscr{A} \cup \{E - X\}$. Since there is no bias extending $\mathscr{A}$, there is no bias extending $\mathscr{A}_1$ or $\mathscr{A}_2$. From our inductive hypothesis there are exact tree-labellings (T_1, α_1) over $\mathscr{A}_1$ and (T_2, α_2) over $\mathscr{A}_2$. A leaf t of T_1 is *bad* if $\alpha_1(t, e) \neq E$ and $\alpha_1(t, e) \cup A \neq E$ for all $A \in \mathscr{A}$, where (t, e) is an incidence, and we define the bad leaves of T_2 similarly. Now if t is a bad leaf of T_1 and (t, e) is an incidence, then $\alpha_1(t, e) \cup X = E$ and so $E - \alpha_1(t, e) \subseteq X$. If $E - \alpha_1(t, e) \neq X$, then from our choice of X, either $E - \alpha_1(t, e) \subseteq A$ for some $A \in \mathscr{A}$ or $\alpha_1(t, e) \subseteq A$ for some $A \in \mathscr{A}$. In the first case $\alpha_1(t, e) \cup A = E$, a contradiction, since t is bad. In the second case $E - X \subseteq \alpha_1(t, e) \subseteq A$, contrary to our choice of X. Thus $E - \alpha_1(t, e) = X$, for every bad leaf t. Since $X \neq \varnothing$, it follows from (3.3) that there is at most one bad leaf in T_1.

On the other hand, we may assume that T_1 has at least one bad leaf, for otherwise (T_1, α_1) is the desired tree-labelling over $\mathscr{A}$. Let t_0 be the unique bad leaf of T_1, incident with an edge e_0. Then $\alpha_1(t_0, e_0) = E - X$. Let the ends of e_0 be t_0, s. Then $\alpha_1(s, e_0) = X$. Since $X \neq E$ and $E - X$ is not a subset of any member of $\mathscr{A}_1$, s is not a leaf of T_1. Let $S = T_1 \backslash t_0$; then s has valency 2 in S.

Let the bad leaves of T_2 be $t_1, ..., t_r$, incident with edges $e_1, ..., e_r$, respectively. Then as before

$$\alpha_2(t_i, e_i) \cup (E - X) = E,$$

that is, $X \subseteq \alpha_2(t_i, e_i)$, for $1 \leqslant i \leqslant r$. Let $S^1, ..., S^r$ be r copies of S, mutually disjoint. For $v \in V(S)$ and $e \in E(S)$ let v^i and e^i denote the corresponding vertex and edge of S^i $(1 \leqslant i \leqslant r)$. Choose $S^1, ..., S^r$ so that $s^i = t_i$ and $V(S^i) \cap V(T_2) = t_i$ $(1 \leqslant i \leqslant r)$, and let T be the tree formed by the union of T_2 and $S^1, ..., S^r$. Every incidence of T is an incidence of exactly one of $T_2, S^1, ..., S^r$. We define α by

$$\alpha(v, e) = \alpha_2(v, e) \qquad \text{if } (v, e) \text{ is an incidence of } T_2$$

$$\alpha(v^i, e^i) = \alpha_1(v, e)\ (1 \leqslant i \leqslant r) \qquad \text{if } (v, e) \text{ is an incidence of } T_1.$$

We claim that (T, α) is a tree-labelling over $\mathscr{A}$, and this follows easily from the fact that

$$\alpha_1(s, e_0) = X \subseteq \alpha_2(t_i, e_i) \qquad (1 \leqslant i \leqslant r).$$

Then the result follows. ∎

In summary then we have shown

(3.5) *The following are equivalent*:

(i) *there is no bias extending* $\mathscr{A}$

(ii) *there is a tree-labelling over* $\mathscr{A}$

(iii) *there is an exact tree-labelling over* $\mathscr{A}$.

We observe also

(3.6) *If there is an exact tree-labelling over* $\mathscr{A}$, *then either* $E = \emptyset$, *or* $E \in \mathscr{A}$, *or there is an exact tree-labelling* (T, α) *over* $\mathscr{A}$ *such that for each leaf* v *and incident edge* e, $\alpha(v, e) \neq E$.

Proof. Choose an exact tree-labelling (T, α) with $|V(T)|$ minimum. Suppose that for some leaf v_0 and incident edge e_0, $\alpha(v_0, e_0) = E$. Let v be the other end of e_0. Then $\alpha(v, e_0) = \varnothing$. If v is also a leaf, then either $E = \varnothing$ or $E \in \mathscr{A}$, as required. We assume then that v has two other neighbours v_1, v_2 in T; let e_i be the edge joining v and v_i $(i = 1, 2)$. Now since (T, α) is exact, $\alpha(v, e_0)$, $\alpha(v, e_1)$, $\alpha(v, e_2)$ are mutually disjoint and have union E. Since $\alpha(v, e_0) = \varnothing$, it follows that $\alpha(v_1, e_1) = \alpha(v, e_2)$ and $\alpha(v, e_1) = \alpha(v_2, e_2)$. Let T' be obtained from T by deleting v and v_0 and adding a new edge f joining v_1 and v_2. We define $\alpha'(v_1, f) = \alpha(v_1, e_1)$, $\alpha'(v_2, f) = \alpha(v_2, e_2)$, and otherwise $\alpha' = \alpha$; then (T', α') is an exact tree-labelling over $\mathscr{A}$ with $|V(T')| < |V(T)|$, a contradiction. ∎

4. Branch-Width

A *branch-decomposition* of a hypergraph G is a pair (T, τ), where T is a ternary tree and τ is a bijection from the set of leaves of T to $E(G)$. The *order* of an edge e of T is the number of vertices v of G such that there are leaves t_1, t_2 of T in different components of $T \backslash e$, with $\tau(t_1)$, $\tau(t_2)$ both incident with v. The *width* of (T, τ) is the maximum order of the edges of T, and the *branch-width* $\beta(G)$ of G is the minimum width of all branch-decompositions of G (or 0 if $|E(G)| \leqslant 1$, when G has no branch-decompositions). For example, Fig. 1 shows a branch-decomposition with width 2 of a series-parallel graph.

Let us prove some properties of branch-width. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges.

(4.1) *If H is a minor of a graph G, then $\beta(H) \leqslant \beta(G)$.*

Proof. We may assume that $|E(H)| \geqslant 2$, for otherwise $\beta(H) = 0$. Let (T, τ) be a branch-decomposition of G with width $\beta(G)$. Let S be a minimal subtree of T such that $\tau^{-1}(e) \in V(S)$ for all $e \in E(H)$, and let T' be obtained from S by suppressing all vertices of valency 2 (that is, for any vertex of valency 2 we delete it and add an edge joining its neighbours and continue this process until no such vertices remain). Let τ' be the restriction of τ to the set of leaves of T'; then (T', τ') is a branch-decomposition of H, and its width is $\leqslant \beta(G)$, as is easily seen. The result follows. ∎

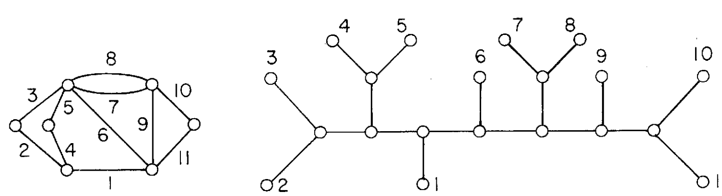

Figure 1

(4.2) *A graph G has branch-width*

(i) 0 *if and only if every component of G has $\leqslant 1$ edge*

(ii) $\leqslant 1$ *if and only if every component of G has $\leqslant 1$ vertex of valency $\geqslant 2$*

(iii) $\leqslant 2$ *if and only if G has no K_4 minor.*

Proof. Statement (i) is clear. The "if" part of (ii) is easy and "only if" follows from (4.1) and the fact that a 2-edge circuit and a 3-edge path both have branch-width 2. The "only if" part of (iii) follows similarly, while the "if" part may be proved by induction on the size of G, using Dirac's theorem [1] that any non-null simple graph with no K_4 minor has a vertex of valency $\leqslant 2$. ∎

The main result of this section is the following. We denote by $\gamma(G)$ the maximum size of an edge of G (setting $\gamma(G)=0$ if $E(G)=\emptyset$). We recall that $\theta(G)$ is the tangle number of G.

(4.3) *For any hypergraph G, $\max(\beta(G), \gamma(G)) = \theta(G)$ unless $\gamma(G)=0$ and $V(G)\neq\emptyset$.*

Proof. Suppose first that $\gamma(G)=0$ and that $\mathscr{T}$ is a tangle in G of order $\geqslant 2$. Choose $(A, B)\in\mathscr{T}$, extreme. By (2.8), $E(B)=\emptyset$, and so $E(A)=E(G)$, contrary to (2.3). Thus, if $\gamma(G)=0$ then $\theta(G)\leqslant 1$. Moreover, if $\gamma(G)=0$ then $\beta(G)=0$, and $\theta(G)=1$ if and only if $V(G)\neq\emptyset$, by (2.5). Thus if $\gamma(G)=0$ the result holds, and we henceforth assume that $\gamma(G)>0$.

Let $E=E(G)$, and for $X\subseteq E$, define $\kappa_0(X)$ to be the number of vertices of G incident both with an edge in X and with an edge in $E-X$. Choose $k\geqslant\gamma(G)$, and let $\kappa(X)=\kappa_0(X)-k$. It is easily seen that κ is a connectivity function, and for every $e\in E(G)$, $\{e\}$ is efficient. Let $\mathscr{A}=\{\{e\}: e\in E(G)\}$.

(1) *There is a bias extending $\mathscr{A}$ if and only if G has a tangle of order $k+1$.*

For if $\mathscr{T}$ is a tangle in G of order $k+1$, let $\mathscr{B}=\{E(A):(A, B)\in\mathscr{T}\}$. Then B is a bias, by (2.3), since $k\geqslant\gamma(G)\geqslant 1$, and it extends $\mathscr{A}$ by the third axiom. For the converse, let $\mathscr{B}$ be a bias extending $\mathscr{A}$, and let $\mathscr{T}$ be the set of all separations (A, B) of G of order $\leqslant k$ with $E(A)\in\mathscr{B}$. We claim that $\mathscr{T}$ is a tangle of order $k+1$. For if (A, B) is a separation of G of order $\leqslant k$, then $E(A)$ and $E(B)$ are both efficient, and so one of $E(A)$, $E(B)$ is in $\mathscr{B}$, $E(A)$, say; but then $(A, B)\in\mathscr{T}$. Thus the first axiom holds, and clearly so does the second. Since $k\geqslant\gamma(G)$ and $\mathscr{B}$ extends $\mathscr{A}$, $(K_e, G\backslash e)\in\mathscr{T}$ for every $e\in E$, where K_e is the hypergraph consisting of e and its ends. By (2.7), $\mathscr{T}$ is a tangle of order $k+1$, as required.

(2) *There is an exact tree-labelling over $\mathscr{A}$ if and only if $\beta(G)\leqslant k$.*

For if $|E| \leqslant 1$, then $\beta(G) = 0 \leqslant k$ and there is an exact tree-labelling over $\mathscr{A}$, and so we may assume that $|E| \geqslant 2$. If (T, τ) is a branch-decomposition of G of width $\leqslant k$, define $\alpha(v, e)$ for each incidence (v, e) to be the set of all edges $\tau(t)$ of G with t and v in different components of $T \backslash e$. Then (T, α) is an exact tree-labelling over $\mathscr{A}$. For the converse, suppose that there is an exact tree-labelling over $\mathscr{A}$. Since $|E| > 1$, it follows that $E \notin \mathscr{A}$ and $E \neq \emptyset$, and so by (3.6) we may choose an exact tree-labelling (T, α) over $\mathscr{A}$ such that for each leaf v and incident edge e, $\alpha(v, e) \neq E$. For such v, e, there exists $\{f\} \in \mathscr{A}$ such that $\alpha(v, e) = E - \{f\}$; we define $f = \tau(v)$. By (3.3), (T, τ) is a branch-decomposition of G of width $\leqslant k$.

From (3.5), (1), and (2) we deduce that

(3) *For all $k \geqslant \gamma(G)$, G has a tangle of order $k+1$ if and only if $k < \beta(G)$.*

Now we deduce the theorem. By (2.4), $\theta(G) \geqslant \gamma(G)$. By setting $k = \theta(G)$ we deduce from (3) that $\beta(G) \leqslant \theta(G)$, and so $\max(\beta(G), \gamma(G)) \leqslant \theta(G)$. By setting $k = \theta(G) - 1$ we deduce from (3) that $\theta(G) \leqslant \max(\beta(G), \gamma(G))$. The result follows. ∎

We apply (4.3) (actually, the easy part of (4.3)) for the following.

(4.4) *For $n \geqslant 0$, K_n has tangle number $\lceil (2/3)\, n \rceil$, and for $n \geqslant 3$, it has branch-width $\lceil (2/3)\, n \rceil$.*

Proof. The result holds for $n \leqslant 3$, and we assume that $n \geqslant 4$. Put $\theta = \lceil (2/3)\, n \rceil$. It is easy to see that K_n has a branch-decomposition of width $\leqslant \theta$. Thus the result follows from (4.3) if we can find a tangle of order θ. Let $\mathscr{T}$ be the set of all separations (A, B) of $G = K_n$ with $|V(A)| < \theta$. If (A, B) is any separation of G then one of $V(A)$, $V(B)$ equals $V(G)$, and so its order equals the smaller of $|V(A)|$, $|V(B)|$. Hence if (A, B) has order $< \theta$ then $\mathscr{T}$ contains one of (A, B), (B, A), and the first axiom is satisfied. For the second axiom, suppose that $(A_i, B_i) \in \mathscr{T}$ $(1 \leqslant i \leqslant 3)$ and $A_1 \cup A_2 \cup A_3 = G$. Since

$$|V(A_1)| + |V(A_2)| + |V(A_3)| \leqslant 3\theta - 3 < 2n$$

some vertex v of G is in at most one of $V(A_1)$, $V(A_2)$, $V(A_3)$; $v \notin V(A_1) \cup V(A_2)$, say. Since $|V(A_3)| < \theta < n$ some vertex u of G is not in $V(A_3)$. But then the edge joining u and v is in none of $E(A_1)$, $E(A_2)$, $E(A_3)$, a contradiction. Thus the second axiom is satisfied. For the third, let $e \in E(G)$, and let K be the graph formed by e and its ends; then $(K, G \backslash e) \in \mathscr{T}$ by definition of $\mathscr{T}$, since $\theta \geqslant 3$, and so $\mathscr{T}$ is a tangle by (2.7). This completes the proof. ∎

Let us mention the following weakening of the second tangle axiom.

(4.5) *Let* $\theta \geqslant 2$, *and let* $\mathscr{T}$ *be a set of separations of a hypergraph* G, *each of order* $< \theta$. *Suppose that the first tangle axiom holds, and*

(i) *if* $(A_1, B_1), (A_2, B_2) \in \mathscr{T}$ *then* $B_1 \not\subseteq A_2$

(ii) *there do not exist subhypergraphs* $A_1, A_2, A_3 \subseteq G$, *mutually edge-disjoint, with* $A_1 \cup A_2 \cup A_3 = G$ *and with* $(A_1, A_2 \cup A_3)$, $(A_2, A_3 \cup A_1)$, $(A_3, A_1 \cup A_2)$ *all in* $\mathscr{T}$.

Then the second tangle axiom holds.

Proof. Suppose that the second axiom fails, and choose (A_1, B_1), (A_2, B_2), $(A_3, B_3) \in \mathscr{T}$ such that $A_1 \cup A_2 \cup A_3 = G$, satisfying

(1) $\sum_{1 \leqslant i \leqslant 3} |V(A_i \cap B_i)|$ *is minimum, and*

(2) *subject to* (1), A_1, A_2, A_3 *are minimal.*

We observe

(3) *For* $1 \leqslant i \leqslant 3$, *if* $v \in V(A_i \cap B_i)$ *then* v *is incident with an edge of* B_i; *and also with an edge of* A_i, *unless* v *belongs to no other* A_j $(j \neq i)$.

For if v is incident with no edge of B_i then $(A_i, B_i \backslash v)$ is a separation, and it belongs to $\mathscr{T}_1$ by the first axiom and (i), contrary to (1). If v is incident with no edge of A_i then $(A_i \backslash v, B_i)$ is a separation, and it belongs to $\mathscr{T}$, by the first axiom and (i), and so by (1), v belongs to no A_j $(j \neq i)$.

(4) *For* $1 \leqslant i, j \leqslant 3$ *with* $i \neq j$, $A_i \subseteq B_j$.

For let $i = 1, j = 2$, say. The sum of the orders of $(A_1 \cap B_2, B_1 \cup A_2)$ and $(A_1 \cup B_2, B_1 \cap A_2)$ equals the sum of the orders of (A_1, B_1) and (A_2, B_2). If $(A_1 \cap B_2, B_1 \cup A_2)$ has order at most that of (A_1, B_1), then since $(A_1 \cap B_2) \cup A_2 \cup A_3 = G$ and $(A_1 \cap B_2, B_1 \cup A_2) \in \mathscr{T}$ by the first axiom and (i), it follows from (2) that $A_1 \cap B_2 = A_1$; that is, $A_1 \subseteq B_2$. Thus $E(A_2) \subseteq E(B_1)$. Suppose that $A_2 \not\subseteq B_1$, and choose $v \in V(A_2) - V(B_1)$. Then $v \in V(A_1 \cap A_2)$, and by (3), v is incident with an edge in $E(A_2) \subseteq E(B_1)$; yet $v \notin V(B_1)$, a contradiction. Thus $A_2 \subseteq B_1$. Similarly, if $(A_2 \cap B_1, B_2 \cup A_1)$ has order at most that of (A_2, B_2), then $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$. The result follows, since one of these inequalities must apply.

From (4), $A_1 \cup A_2 \subseteq B_3$, and so $(A_3, A_1 \cup A_2)$ is a separation of order $< \theta$. Since $(A_3, B_3) \in \mathscr{T}$, it follows from (i) that $(A_1 \cup A_2, A_3) \notin \mathscr{T}$, and so $(A_3, A_1 \cup A_2) \in \mathscr{T}$, from the first axiom. Similarly $(A_1, A_2 \cup A_3)$, $(A_2, A_3 \cup A_1) \in \mathscr{T}$, contrary to (ii). ∎

5. Branch-Width and Tree-Width

A *tree-decomposition* of a hypergraph G is a pair (T, τ), where T is a tree and for $t \in V(T)$, $\tau(t)$ is a subhypergraph of G with the following properties:

(i) $\bigcup (\tau(t) : t \in V(T)) = G$

(ii) for distinct $t, t' \in V(T)$, $E(\tau(t) \cap \tau(t')) = \emptyset$

(iii) for $t, t', t'' \in V(T)$, if t' is on the path of T between t and t'' then $\tau(t) \cap \tau(t'') \subseteq \tau(t')$.

The *width* of such a tree-decomposition is the maximum of $(|V(\tau(t))| - 1)$, taken over all $t \in V(T)$, and the *tree-width* $\omega(G)$ of G is the minimum width of all tree-decomposition of G. (Thus, $\omega(G) \geqslant 0$ unless $V(G) = \emptyset$, when $\omega(G) = -1$.)

Let us compare tree-width and branch-width.

(5.1) *For any hypergraph* G, $\max(\beta(G), \gamma(G)) \leqslant \omega(G) + 1 \leqslant \max(\lfloor (3/2)\,\beta(G) \rfloor, \gamma(G), 1)$.

Proof. If $\gamma(G) = 0$ then $\beta(G) = 0$ and $\omega(G) \leqslant 0$, and the result holds. We assume then that $\gamma(G) > 0$, and so $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. If $|E(G)| = 1$ then $\beta(G) = 0$ and $\omega(G) = \gamma(G) - 1$, and again the result holds. Thus we may assume that $|E(G)| \geqslant 2$. Since the removal of isolated vertices does not change any of β, γ, ω, we may assume that there are no isolated vertices in G. We show the second inequality first.

Let (T, τ) be a branch-decomposition of G of width $\beta(G)$. For each $t \in V(T)$ we define a subhypergraph $\sigma(t)$ of G as follows:

(i) if t is a leaf of T, let $\sigma(t)$ be the hypergraph consisting of $\tau(t)$ and its ends

(ii) if t is not a leaf of T, let U_t consist of those vertices v of G for which there are edges f, g of G, both incident with v, such that t lies on the path of T between $\tau^{-1}(f)$ and $\tau^{-1}(g)$. Let $V(\sigma(t)) = U_t$, $E(\sigma(t)) = \emptyset$.

It is easy to verify that (T, σ) is a tree-decomposition of G. Let us bound its width. If t is a leaf of T, $|V(\sigma(t))| \leqslant \gamma(G)$. If t is not a leaf of T, let e_1, e_2, e_3 be the three edges of T incident with t. For any $v \in U_t$, v contributes to the order of at least two of e_1, e_2, e_3, and so $2\,|U_t| \leqslant 3\beta(G)$. Thus, this tree-decomposition has width $\leqslant \max(\gamma(G), (3/2)\,\beta(G)) - 1$, and so $\omega(G) + 1 \leqslant \max(\gamma(G), (3/2)\,\beta(G))$, as required.

Now we show the first inequality. Clearly $\gamma(G) \leqslant \omega(G) + 1$. Let (T, τ) be a tree-decomposition of G of width $\omega(G)$.

(1) *We may assume that for each* $e \in E(G)$, *there is a leaf* t *of* T *with* $E(\tau(t)) = \{e\}$ *and* $V(\tau(t))$ *the set of ends of* e, *and hence that* $E(\tau(t)) = \emptyset$ *for each* $t \in V(T)$ *with valency* $\geqslant 2$.

For if for some e there is no such t, we choose $t' \in V(T)$ with $e \in E(\tau(t'))$; we add a new vertex t to T adjacent only to t'; we remove e from $\tau(t')$, and define $\tau(t)$ to be the hypergraph formed by e and its ends. This provides a new tree-decomposition of G of width $\omega(G)$. By continuing this process we may arrange that (1) holds.

(2) *We may assume that* $|E(\tau(t))| = 1$ *for each leaf* t *of* T.

For by (1), $|E(\tau(t))| \leqslant 1$. If $E(\tau(t)) = \emptyset$ let T' be obtained from T by deleting t, and let τ' be the restriction of τ to $V(T')$; then since G has no isolated vertices it follows that (T', τ') is a new tree-decomposition of G of width $\omega(G)$ still satisfying (1). By continuing this process we may arrange that (2) holds.

(3) *We may assume that every vertex of* T *has valency* $\leqslant 3$.

For if $t \in V(T)$ has valency $\geqslant 4$, we may choose a tree T' and an edge f of T' such that T is obtained from T' by contracting f, and the two ends t_1, t_2 of f both have valency less than the valency of t, and we define $\tau(t_1) = \tau(t_2) = \tau(t)$. The new tree-decomposition still has width $\omega(G)$ and still satisfies (1) and (2), and by repeating this process we may arrange that (3) holds.

Now let $E(\tau(t)) = \{\sigma(t)\}$ for each leaf t of T. Let S be the tree obtained from T by suppressing each vertex of valency 2. Then (S, σ) is a branch-decomposition of G. For $f \in E(S)$, the order of f in (S, σ) is at most the number of vertices in $\tau(t)$, where t is an end of f, and hence at most $\omega(G)+1$. Thus $\beta(G) \leqslant \omega(G)+1$, as required. ∎

Incidentally, both extremes of (5.1) can occur. For if $G = K_n$ (for some $n > 0$ divisible by 3) then $\omega(G) = \lfloor (3/2)\,\beta(G) \rfloor - 1$, by (4.4), since $\omega(G) = n-1$, while if G is obtained from $K_{n,n}$ by deleting a perfect matching (for some $n \geqslant 4$) then it can be shown that $\omega(G) = n-1$ and $\beta(G) = n$.

We deduce

(5.2) *For any hypergraph* G, $\theta(G) \leqslant \omega(G) + 1 \leqslant (3/2)\,\theta(G)$.

Proof. For from (5.1),

$$\max(\beta(G), \gamma(G)) \leqslant \omega(G) + 1 \leqslant \max(\tfrac{3}{2}\beta(G), \gamma(G), 1)$$

and from (4.3), $\max(\beta(G), \gamma(G)) = \theta(G)$ unless $\gamma(G) = 0$ and $V(G) \neq \emptyset$.

Moreover the proof of (2.5) shows that $\theta(G) \geqslant 1$ unless $V(G) = \varnothing$. Thus if $\gamma(G) \neq 0$ and hence $V(G) \neq \varnothing$, then

$$\theta(G) = \max(\beta(G), \gamma(G)) \leqslant \omega(G) + 1 \leqslant \max(\tfrac{3}{2}\beta(G), \gamma(G), 1)$$
$$= \tfrac{3}{2}\max(\beta(G), \gamma(G)) = \tfrac{3}{2}\theta(G),$$

as required. If $\gamma(G) = 0$ and $V(G) \neq \varnothing$, then $\omega(G) = 0$ and $\theta(G) = 1$, and the result holds. Finally, if $V(G) = \varnothing$, then $\theta(G) = 0$ and $\omega(G) = -1$, and again the result holds. ∎

6. New Tangles from Old

The object of this section is to provide some operations on tangles. The simplest is the following. Let $\mathscr{T}$ be a tangle of order θ in a hypergraph G, let $1 \leqslant \theta' \leqslant \theta$, and let $\mathscr{T}'$ be the set of all members of $\mathscr{T}$ with order $< \theta'$. Then it is easy to see that $\mathscr{T}'$ is a tangle in G of order θ'; we call $\mathscr{T}'$ the *truncation* of $\mathscr{T}$ to order θ'. We observe also that if $\mathscr{T}, \mathscr{T}'$ are tangles in G then $\mathscr{T}' \subseteq \mathscr{T}$ if and only if $\mathscr{T}'$ is a truncation of $\mathscr{T}$.

For graphs G, a second construction extends a tangle in a minor of G to a tangle in G, as follows.

(6.1) *Let H be a minor of a graph G, and let $\mathscr{T}'$ be a tangle in H of order $\theta \geqslant 2$. Let $\mathscr{T}$ be the set of all separations (A, B) of G of order $< \theta$ such that there exists $(A', B') \in \mathscr{T}'$ with $E(A') = E(A) \cap E(H)$. Then $\mathscr{T}$ is a tangle in G of order θ.*

Proof. We must verify the three axioms. First, let (A, B) be a separation of G of order $< \theta$. Then we may choose a separation (A', B') of H' such that $E(A') = E(A) \cap E(H)$, and every vertex of $V(A' \cap B')$ is incident with an edge of $E(A')$ and with an edge of $E(B')$. Then (A', B') has order at most the order of (A, B) and so $< \theta$; thus, $\mathscr{T}'$ contains one of (A', B'), (B', A'), and so $\mathscr{T}$ contains one of (A, B), (B, A).

For the second axiom, suppose that $(A_i, B_i) \in \mathscr{T}$ $(1 \leqslant i \leqslant 3)$ with $A_1 \cup A_2 \cup A_3 = G$, and let $(A_i', B_i') \in \mathscr{T}'$ $(1 \leqslant i \leqslant 3)$ be the corresponding separations of H. Then $E(A_1' \cup A_2' \cup A_3') = E(H)$, contrary to (2.3). Finally, it is clear from (2.7) that the third axiom holds. ∎

We call $\mathscr{T}$ in (6.1) the tangle in G *induced* by $\mathscr{T}'$.

A third construction reverses this process. Let G be a hypergraph and W a set. We denote by G/W the hypergraph G' with vertex set $V(G) - W$ and edge set $E(G)$, in which $v \in V(G')$ and $e \in E(G')$ are incident if and only if they are incident in G. (This may produce edges with no ends.)

(6.2) *Let $\mathscr{T}$ be a tangle of order θ in a hypergraph G, and let $W \subseteq V(G)$ with $|W| < \theta$. Let $\mathscr{T}'$ be the set of all separations (A', B') of G/W such that there exists $(A, B) \in \mathscr{T}$ with $W \subseteq V(A \cap B)$, $A/W = A'$, and $B/W = B'$. Then $\mathscr{T}'$ is a tangle in G/W of order $\theta - |W|$.*

Proof. Certainly every member of $\mathscr{T}'$ has order $< \theta - |W|$. For any separation (A', B') of G/W of order $< \theta - |W|$, there is a separation (A, B) of G of order $< \theta$ with $W \subseteq V(A \cap B)$, $A/W = A'$, and $B/W = B'$, and since $\mathscr{T}$ contains one of (A, B), (B, A), it follows that $\mathscr{T}'$ contains one of (A', B'), (B', A'). Thus the first axiom is satisfied.

For the second, suppose that $(A'_i, B'_i) \in \mathscr{T}'$ $(1 \leqslant i \leqslant 3)$. Choose $(A_i, B_i) \in \mathscr{T}$ with $W \subseteq V(A_i \cap B_i)$, $A_i/W = A'_i$, and $B_i/W = B'_i$ $(1 \leqslant i \leqslant 3)$. Since $A_1 \cup A_2 \cup A_3 \neq G$, it follows that $A'_1 \cup A'_2 \cup A'_3 \neq G/W$, and hence the second axiom holds.

For the third, let $(A', B') \in \mathscr{T}'$. Choose $(A, B) \in \mathscr{T}$ with $W \subseteq V(A \cap B)$, $A/W = A'$, and $B/W = B'$. Then $V(A) \neq V(G)$, and so $V(A') \neq V(G/W)$, as required. ∎

We denote the tangle $\mathscr{T}'$ of (6.2) by $\mathscr{T}/W$. We observe

(6.3) *Let $\mathscr{T}$, θ, G, W be as in (6.2), and let (A, B) be a separation of G. Then $(A/W, B/W) \in \mathscr{T}/W$ if and only if $(A, B) \in \mathscr{T}$ and $|V(A \cap B) - W| < \theta - |W|$.*

Proof. Let A^+ be a subhypergraph of G with $V(A^+) = V(A) \cup W$ and $E(A^+) = E(A)$, and define B^+ similarly. Then (A^+, B^+) is a separation of G, $W \subseteq V(A^+ \cap B^+)$, $A^+/W = A/W$, and $B^+/W = B/W$. By definition of $\mathscr{T}/W$, $(A^+, B^+) \in \mathscr{T}$ if and only if $(A/W, B/W) \in \mathscr{T}/W$. But by (2.9), $(A^+, B^+) \in \mathscr{T}$ if and only if $|V(A^+ \cap B^+)| < \theta$ and $(A, B) \in \mathscr{T}$. Since $|V(A^+ \cap B^+)| = |W| + |V(A \cap B) - W|$, the result follows. ∎

7. A Tangle in a Grid

Let $\theta \geqslant 2$ be an integer. Let G be a simple graph with $V(G) = \{(i, j): 1 \leqslant i, j \leqslant \theta\}$, where (i, j) and (i', j') are adjacent if $|i' - i| + |j' - j| = 1$. We call G a θ-grid. The object of this section is to prove the existence of a natural tangle of order θ in a θ-grid.

Let G be the θ-grid defined above. For $1 \leqslant i \leqslant \theta$, let P_i be the path of G with vertex set $\{(i, j): 1 \leqslant j \leqslant \theta\}$, and for $1 \leqslant j \leqslant \theta$, define Q_j similarly. When $X \subseteq E(G)$, we define $\partial(X)$ to be the set of vertices $v \in X$ such that v is incident with an edge in X and with an edge in $E(G) - X$.

(7.1) *If $X \subseteq E(G)$ and $|\partial(X)| < \theta$ then X includes $E(P_i)$ for some i $(1 \leqslant i \leqslant \theta)$ if and only if X includes $E(Q_j)$ for some j $(1 \leqslant j \leqslant \theta)$.*

Proof. Suppose that $E(P_i) \subseteq X$ for some i $(1 \leqslant i \leqslant \theta)$. Then $V(Q_j)$ contains an end of an edge in X for $1 \leqslant j \leqslant \theta$, since each Q_j meets P_i. But not every Q_j meets $\partial(X)$, since $|\partial(X)| < \theta$, and so for some j $(1 \leqslant j \leqslant \theta)$, $E(Q_j) \subseteq X$, as required. ∎

If $X \subseteq E(G)$, we say that X is *small* (in G) if $|\partial(X)| < \theta$ and X includes $E(P_i)$ for no i $(1 \leqslant i \leqslant \theta)$. The following is the main lemma used to obtain the required tangle, and we are grateful to D. Kleitman and M. Saks for finding the proof.

(7.2) *If G is a θ-grid and $X_1, X_2, X_3 \subseteq E(G)$ with $X_1 \cup X_2 \cup X_3 = E(G)$, then not all of X_1, X_2, X_3 are small in G.*

Proof. We proceed by induction on θ. If $\theta = 2$ the result is trivial, and so we assume that $\theta \geqslant 3$ and that the result is true for $\theta - 1$. Let $P_1, ..., P_\theta$, $Q_1, ..., Q_\theta$ be as before.

If $E(Q_j) \subseteq X_1$, X_2, or X_3 for some j, the result is true by (7.1). Thus we may assume that each $E(Q_j)$ meets at least two of X_1, X_2, X_3, and in particular, without loss of generality, that

$$E(Q_\theta) \cap X_1 \neq \emptyset \neq E(Q_\theta) \cap X_2.$$

We suppose that all of X_1, X_2, X_3 are small. Thus, for $1 \leqslant j \leqslant \theta$ and $1 \leqslant k \leqslant 3$, if $E(Q_j)$ meets X_k, then $V(Q_j)$ meets $\partial(X_k)$. Moreover, if both ends of Q_j are incident with edges in X_k, then $|V(Q_j) \cap \partial(X_k)| \geqslant 2$. Now suppose that neither $E(P_1)$ nor $E(P_\theta)$ meets X_3. Then for $1 \leqslant j \leqslant \theta$ both ends of Q_j are incident with edges in $X_1 \cup X_2$. From the above remarks, we deduce that

$$|V(Q_j) \cap \partial(X_1)| + |V(Q_j) \cap \partial(X_2)| \geqslant 2.$$

By summing over j, we find that $|\partial(X_1)| + |\partial(X_2)| \geqslant 2\theta$, a contradiction. Thus one of $E(P_1)$, $E(P_\theta)$, say $E(P_\theta)$, meets X_3. Hence $E(P_\theta \cup Q_\theta)$ meets each of X_1, X_2, X_3 and hence $V(P_\theta \cup Q_\theta)$ meets each of $\partial(X_1)$, $\partial(X_2)$, $\partial(X_3)$.

Put $G' = G \backslash V(P_\theta \cup Q_\theta)$. Then G' is a $(\theta - 1)$-grid. Put $X'_k = X_k \cap E(G')$ $(1 \leqslant k \leqslant 3)$. Then $X'_1 \cup X'_2 \cup X'_3 = E(G')$. Let ∂' be the ∂ function in G'. Now

$$\partial'(X'_k) \subset \partial(X_k) \qquad (1 \leqslant k \leqslant 3)$$

since $V(P_\theta \cup Q_\theta)$ meets $\partial(X_k)$, and so

$$|\partial'(X'_k)| \leqslant \theta - 2 \qquad (1 \leqslant k \leqslant 3).$$

By our inductive hypothesis, one of X'_1, X'_2, X'_3 is not small in G'. By (7.1), we may choose i', j' with $1 \leqslant i', j' \leqslant \theta - 1$, and $1 \leqslant k \leqslant 3$ such that

$$E((P_{i'} \cup Q_{j'}) \cap G') \subseteq X'_k.$$

If $k = 1$ or 2, then every $V(Q_j)$ contains an end of an edge in X_k $(1 \leqslant j \leqslant \theta)$; for if $j = \theta$, this was shown earlier, and if $j < \theta$, then $V(Q_j)$ meets $V(P_{i'})$. Hence each $V(Q_j)$ meets $\partial(X_k)$, and so $|\partial(X_k)| \geqslant \theta$, a contradiction. Similarly, if $k = 3$, then every $V(P_i)$ meets $\partial(X_k)$, and again we have a contradiction. This completes the proof. ∎

From (7.2) we may infer the existence of the desired tangle. Given a θ-grid G with $P_1, ..., P_\theta$, $Q_1, ..., Q_\theta$ as before, let $\mathscr{T}$ be the set of all separations (A, B) of G of order $< \theta$ such that $E(A)$ is small.

(7.3) *$\mathscr{T}$ is a tangle in G of order θ.*

Proof. Let (A, B) be a separation of G of order $< \theta$. Suppose that neither $E(A)$ nor $E(B)$ is small. Choose h, i with $1 \leqslant h$, $i \leqslant \theta$ such that $E(P_h) \subseteq E(A)$ and $E(P_i) \subseteq E(B)$. Thus $V(P_h) \subseteq V(A)$ and $V(P_i) \subseteq V(B)$. For $1 \leqslant j \leqslant \theta$, $\emptyset \neq V(Q_j \cap P_h) \subseteq V(Q_j \cap A)$, and similarly $V(Q_j \cap B) \neq \emptyset$, and so $V(Q_j \cap A \cap B) \neq \emptyset$ since (A, B) is a separation. But then $|V(A \cap B)| \geqslant \theta$, a contradiction. Thus one of $E(A)$, $E(B)$ is small, and so $\mathscr{T}$ satisfies the first axiom. That $\mathscr{T}$ is a tangle then follows from (7.2). ∎

The following was shown in [3].

(7.4) *For every $\theta \geqslant 2$ there exists $\phi \geqslant 0$ such that every graph with tree-width $\geqslant \phi$ has a θ-grid minor.*

Since any graph with a θ-grid minor has tree-width $\geqslant \theta$, one can say, roughly, that a graph has large tree-width if and only if it has a large grid minor. But (5.2) tells us that a graph has large tree-width if and only if it has a tangle of large order. One might therefore hope for a direct connection between tangles and grid minors, not via tree-width. The connection in one direction is easy, as follows. Let H be a minor of G, isomorphic to a θ-grid. Then the tangle in H described in (7.3) induces a tangle $\mathscr{T}$ in G of order θ, by (6.1). A kind of converse is provided by the following strengthening of (7.4), proved in [7].

(7.5) *For every $\theta \geqslant 2$ there exists $\phi \geqslant \theta$ such that for every graph G and every tangle $\mathscr{T}$ in G of order $\geqslant \phi$, the truncation of $\mathscr{T}$ to order θ is the tangle induced by some θ-grid minor of G.*

8. Robust and Titanic Separations

The object of this section is to prove a technical lemma for use in a later paper. A separation (A, B) of G is *robust* if for every separation (C, D) of A, one of the separations $(C, B \cup D)$, $(D, B \cup C)$ has order at least that of (A, B). (Incidentally, Noga Alon (unpublished) has shown that deciding if a separation is robust is NP-complete.) We need the following lemma.

(8.1) *Let (A, B) be a robust separation of G, and let (C, D) be a separation of G. Then one of $(A \cup C, B \cap D)$, $(A \cup D, B \cap C)$ has order at most that of (C, D).*

Proof. Now $(A \cap C, A \cap D)$ is a separation of A. Since (A, B) is robust, we may assume (exchanging C, D if necessary) that

$$|V((A \cap C) \cap (B \cup D))| = |V((A \cap C) \cap (B \cup (A \cap D)))| \geqslant |V(A \cap B)|.$$

But

$$|V(A \cap B)| + |V(C \cap D)|$$
$$= |V((A \cap C) \cap (B \cup D))| + |V((A \cup C) \cap (B \cap D))|,$$

and the result follows. ∎

A separation (A, B) of G is *titanic* if for every triple (X, Y, Z) of subhypergraphs of A such that $A = X \cup Y \cup Z$ and $E(X)$, $E(Y)$, $E(Z)$ are mutually disjoint, we have either

$$|V((X \cup Y) \cap Z)| \geqslant |V((X \cup Y) \cap B)|$$

or

$$|V((Y \cup Z) \cap X)| \geqslant |V(Y \cup Z) \cap B)|$$

or

$$|V((Z \cup X) \cap Y)| \geqslant |V((Z \cup X) \cap B)|.$$

(8.2) *Every titanic separation is robust.*

Proof. Let (A, B) be a titanic separation, and let (C, D) be a separation of A. Put $X = C$, $Y = D$, and let Z be the hypergraph with $V(Z) = E(Z) = \emptyset$. Since (A, B) is titanic, we deduce that either $0 \geqslant |V(A \cap B)|$ or $|V(C \cap D)| \geqslant |V(B \cap D)|$ or $|V(C \cap D)| \geqslant |V(B \cap C)|$. If $V(A \cap B) = \emptyset$ then (A, B) is robust. Thus, by symmetry, we may assume that $|V(B \cap C)| \leqslant |V(C \cap D)|$. But

$$|V(A \cap B)| = |V(B \cap C)| + |V(B \cap D) - V(C)|$$

and

$$|V((B \cup C) \cap D)| = |V(C \cap D)| + |V(B \cap D) - V(C)|,$$

and so $|V(A \cap B)| \leqslant |V((B \cup C) \cap D)|$, as required. ∎

The main result of this section is another way to construct new tangles from old, the following.

(8.3) *Let (C, D) be a separation of a hypergraph G, and let (C', D) be a titanic separation of a hypergraph G', with $V(C \cap D) = V(C' \cap D)$. Let $\mathscr{T}$ be a tangle in G of order $\theta \geqslant 2$ with $(C, D) \in \mathscr{T}$. Let $\mathscr{T}'$ be the set of all separations (A', B') of G' of order $< \theta$ such that there exists $(A, B) \in \mathscr{T}$ with $E(A \cap D) = E(A' \cap D)$. Then $\mathscr{T}'$ is a tangle in G' of order θ.*

Proof. We verify the hypotheses of (4.5). For the first axiom, let (A', B') be a separation of G' of order $< \theta$. Since (C', D) is robust by (8.2), we may assume by (8.1) (exchanging A', B' if necessary) that $(A' \cap D, B' \cup C')$ has order at most that of (A', B'). Now $(A' \cap D, (B' \cap D) \cup C)$ is a separation of G with the same order as $(A' \cap D, B' \cup C')$, since $B' \cup C' = (B' \cap D) \cup C'$ and

$$(A' \cap D) \cap C = A' \cap (D \cap C) = A' \cap (D \cap C') = (A' \cap D) \cap C'.$$

Hence $(A' \cap D, (B' \cap D) \cup C)$ has order $< \theta$ and so $\mathscr{T}$ contains one of $(A' \cap D, (B' \cap D) \cup C)$, $((B' \cap D) \cup C, A' \cap D)$. If the first, then $(A', B') \in \mathscr{T}'$, while if the second, then since $E(((B' \cap D) \cup C) \cap D) = E(B' \cap D)$, it follows that $\mathscr{T}'$ contains (B', A'). This verifies that $\mathscr{T}'$ satisfies the first axiom.

For (4.5) (i), suppose that $(A'_1, B'_1), (A'_2, B'_2) \in \mathscr{T}'$. Choose $(A_i, B_i) \in \mathscr{T}$ with $E(A_i \cap D) = E(A'_i \cap D)$ $(i = 1, 2)$. Since $(C, D) \in \mathscr{T}$, $E(C \cup A_1 \cup A_2) \neq E(G)$ by (2.3), and so $E(D) \nsubseteq E(A_1 \cup A_2)$. Hence $E(D) \nsubseteq E(A'_1 \cup A'_2)$, and so $A'_1 \cup A'_2 \neq G'$, and $B'_1 \nsubseteq A'_2$, as required.

For (4.5) (ii), suppose that A'_1, A'_2, A'_3 are mutually edge-disjoint subhypergraphs of G' with union G', and $(A'_i, B'_i) \in \mathscr{T}'$ for $i = 1, 2, 3$, where $B'_1 = A'_2 \cup A'_3$, $B'_2 = A'_3 \cup A'_1$, $B'_3 = A'_1 \cup A'_2$. Choose $(A_i, B_i) \in \mathscr{T}$ with $E(A_i \cap D) = E(A'_i \cap D)$ $(1 \leqslant i \leqslant 3)$. Let $F_i = A'_i \cap C'$ $(1 \leqslant i \leqslant 3)$. Then $F_1 \cup F_2 \cup F_3 = C'$, and since (C', D) is titanic we may renumber so that

$$|V((F_2 \cup F_3) \cap F_1)| \geqslant |V((F_2 \cup F_3) \cap D)|;$$

that is,

$$|V(B'_1 \cap C' \cap A'_1)| \geqslant |V(B'_1 \cap C' \cap D)|.$$

Now $V(A_1' \cup C') = V(C') \cup (V(A_1') - V(C'))$, and so

$$|V((A_1' \cup C') \cap (B_1' \cap D))| = |V(B_1' \cap C' \cap D)| + |(V(A_1') - V(C')) \cap V(B_1' \cap D)|.$$

Moreover, since $V(A_1' \cap B_1') - V(C') = (V(A_1') - V(C')) \cap V(B_1' \cap D)$, it follows that

$$|V(A_1' \cap B_1')| = |V(B_1' \cap C' \cap A_1')| + |(V(A_1') - V(C')) \cap V(B_1' \cap D)|.$$

We deduce that $(A_1' \cup C', B_1' \cap D)$ has order at most that of (A_1', B_1') and hence $<\theta$. It follows that $((A_1' \cap D) \cup C, B_1' \cap D)$ is a separation of G of order $<\theta$, and so $\mathscr{T}$ contains one of $(B_1' \cap D, (A_1' \cap D) \cup C)$, $((A_1' \cap D) \cup C, B_1' \cap D)$. The first is impossible by (2.3), since (C, D), $(A_1, B_1) \in \mathscr{T}$ and

$$E((B_1' \cap D) \cup C \cup A_1) = E(G).$$

The second is impossible by (2.3), since (A_2, B_2), $(A_3, B_3) \in \mathscr{T}$ and

$$E((A_1' \cap D) \cup C \cup A_2 \cup A_3) = E(G).$$

This contradiction completes the verification of (4.5) (ii). Thus, from (4.5), we deduce that $\mathscr{T}'$ satisfies the second axiom.

To verify the third axiom, we verify the hypothesis of (2.7). Let $e \in E(G')$ with size $<\theta$, and let K_e be as in (2.7). If $e \in E(D)$, then since $(K_e, G \backslash e) \in \mathscr{T}$ by (2.7) applied to G, $\mathscr{T}$, it follows from the definition of $\mathscr{T}'$ that $(K_e, G' \backslash e) \in \mathscr{T}'$. If $e \in E(C')$, then since $(C, D) \in \mathscr{T}$ and $E(C \cap D) = E(K_e \cap D)$, it again follows that $(K_e, G' \backslash e) \in \mathscr{T}'$ from the definition of $\mathscr{T}'$. Thus, from (2.7), we deduce that $\mathscr{T}'$ satisfies the third axiom, as required. ∎

As an application, we observe

(8.4) *Let $\mathscr{T}$ be a tangle of order $\theta \geq 2$ in a hypergraph G, and let $e \in E(G)$ with at most one end. Let $\mathscr{T}'$ be the set of all separations (A', B') of $G \backslash e$ of order $<\theta$ such that there exists $(A, B) \in \mathscr{T}$ with $E(A \cap (G \backslash e)) = E(A')$. Then $\mathscr{T}'$ is a tangle in $G \backslash e$ of order θ.*

Proof. Let C be the subhypergraph of G formed by e and its ends and let $C' = C \backslash e$ and $D = G \backslash e$. Then $(C, D) \in \mathscr{T}$ and (C', D) is titanic, as is easily seen, and the result follows from (8.3). ∎

Thus, if we delete all edges of G with ≤ 1 end, we do not change its tangle number. (This holds even for tangle number ≤ 1, as is easily seen.) (8.4) has the following consequence.

(8.5) *Let $\mathscr{T}$ be a tangle in a graph G of order $\theta \geqslant 1$. Let $W \subseteq V(G)$ with $|W| < \theta$. Let $\mathscr{T}'$ be the set of all separations (A', B') of $G \backslash W$ of order $< \theta - |W|$ such that there exists $(A, B) \in \mathscr{T}$ with $W \subseteq V(A \cap B)$ and $A \backslash W = A'$, $B \backslash W = B'$. Then $\mathscr{T}'$ is a tangle in $G \backslash W$ of order $\theta - |W|$.*

Proof. Since $|W| < \theta$, the result is obvious when $\theta = 1$, and so we may assume that $\theta \geqslant 2$. Now $G \backslash W$ is obtained from G/W by deleting edges with at most one end, and $\mathscr{T}'$ is obtained from $\mathscr{T}/W$ by repeating the operation of (8.4). The result follows. ∎

9. Laminar Separations

We have seen in (5.2) that the tangles of large order are obstructions to the existence of tree-decompositions of small width. Our next result is a counterpart of this, that there is a tree-decomposition into pieces which correspond to the tangles.

Let (A_1, B_1), (A_2, B_2) be separations of a hypergraph G. We say these separations *cross* unless either $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$, or $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$, or $B_1 \subseteq A_2$ and $B_2 \subseteq A_1$, or $B_1 \subseteq B_2$ and $A_2 \subseteq A_1$. A set of separations is *laminar* if no two of its members cross.

Let (T, τ) be a tree-decomposition of a hypergraph G. For each $e \in E(T)$, let T_1, T_2 be the components of $T \backslash e$ and let

$$G_i^e = \bigcup (\tau(t) : t \in V(T_i)) \qquad (i = 1, 2).$$

Then (G_1^e, G_2^e) is a separation of G, and we call (G_1^e, G_2^e) and (G_2^e, G_1^e) the separations *made* by e (under the given tree-decomposition).

(9.1) *If (T, τ) is a tree-decomposition of G, then the set of all separations of G made by edges of T is laminar. Conversely, if $\{(A_i, B_i) : 1 \leqslant i \leqslant k\}$ is a laminar set of separations of G, there is a tree-decomposition (T, τ) of G such that*

(i) *for $1 \leqslant i \leqslant k$, (A_i, B_i) is made by a unique edge of T*

(ii) *for each edge e of T, at least one of the two separations made by e equals (A_i, B_i) for some i $(1 \leqslant i \leqslant k)$.*

The proof is easy and is left to the reader.

We wish to arrange a "tie-breaking" mechanism so that no two distinct separations are counted as having the same order (except for reversal). A *tie-breaker* λ in a hypergraph G is a function from the set of all separations of G into some linearly ordered set $(\Lambda, <)$, satisfying certain axioms given below. For each separation (A, B), $\lambda(A, B)$ is called the *λ-order* of (A, B), and, if (A, B), (C, D) are separations, we say that (A, B) has *smaller λ-order* than (C, D) if $\lambda(A, B) < \lambda(C, D)$. The tie-breaker λ must satisfy the following conditions:

(i) if (A, B), (C, D) are separations of G, they have the same λ-order if and only if $(A, B) = (C, D)$ or $(A, B) = (D, C)$

(ii) if (A, B), (C, D) are separations of G, then either $(A \cup C, B \cap D)$ has λ-order at most that of (A, B) or $(A \cap C, B \cup D)$ has λ-order smaller than that of (C, D)

(iii) if (A, B), (C, D) are separations of G and (A, B) has smaller order than (C, D), then (A, B) has smaller λ-order than (C, D).

We refer to these as the *first*, *second*, and *third tie-breaker axioms.*

(9.2) *In every hypergraph there is a tie-breaker.*

Proof. Let $(A, <)$ be the set of all triples of real numbers, ordered lexicographically; thus, $(a, b, c) < (a', b', c')$ if $a < a'$, or $a = a'$ and $b < b'$, or $a = a'$ and $b = b'$ and $c < c'$. For any hypergraph G, let $L(G) = V(G) \cup E(G)$. Let G be a hypergraph. Choose a function μ from $L(G) \times L(G)$ into the set of positive real numbers such that

(i) $\mu(x, y) = \mu(y, x)$ for all $x, y \in L(G)$, and

(ii) for every choice of rationals $\alpha(x, y)$ $(x, y \in L(G))$ such that $\sum_{x,y} \alpha(x, y)\, \mu(x, y) = 0$, we have $\alpha(x, y) = -\alpha(y, x)$ for all $x, y \in L(G)$.

For each separation (A, B) of G, define $\lambda(A, B) = (N_1, N_2, N_3)$, where

$$N_1 = |V(A \cap B)|$$

$$N_2 = \sum (\mu(x, x) : x \in V(A \cap B))$$

$$N_3 = \sum (\mu(x, y) : x \in L(A) - L(B), y \in L(B) - L(A)).$$

(1) *If (A, B) and (A', B') are separations of G with the same λ-order then $(A', B') = (A, B)$ or (B, A).*

For let (A, B) have λ-order (N_1, N_2, N_3), and let (A', B') have λ-order (N_1', N_2', N_3'). Let $V(A \cap B) = Z$, $L(A) - L(B) = X$, $L(B) - L(A) = Y$, and define Z', X', Y' similarly. Then (X, Y, Z), (X', Y', Z') are partitions of $L(G)$, and we must show that $Z' = Z$ and that $(X', Y') = (X, Y)$ or (Y, X). Now since $N_2 = N_2'$,

$$\sum_{x \in Z} \mu(x, x) = \sum_{x \in Z'} \mu(x, x),$$

and so $Z = Z'$ from (ii) above. Moreover, since $N_3 = N'_3$,

$$\sum (\mu(x, y) : x \in X, y \in Y) = \sum (\mu(x, y) : x \in X', y \in Y').$$

Hence

$$\{\{x, y\} : x \in X, y \in Y\} = \{\{x, y\} : x \in X', y \in Y'\},$$

and the claim follows.

(2) *Let (A, B), (C, D) be separations of G. Then so are $(A \cup C, B \cap D)$, $(A \cap C, B \cup D)$, and the sum of their λ-orders is at most the sum of the λ-orders of (A, B), (C, D).*

This follows by comparing (for each $x, y \in L(G)$) the number of occurrences of $\mu(x, y)$ and $\mu(y, x)$ in the expressions for the λ-orders of (A, B) and (C, D) with the corresponding numbers for the other two separations.

From (1) and (2), it follows that the first and second tie-breaker axioms are satisfied, and clearly so is the third, as required. ∎

The following strengthening of the second axiom is sometimes useful.

(9.3) *Let λ be a tie-breaker in a hypergraph G, and let (A, B), (C, D) be separations of G. Then either*

(i) *$(A \cup C, B \cap D)$ has smaller λ-order than (A, B), or*
(ii) *$(A \cap C, B \cup D)$ has smaller λ-order than (C, D), or*
(iii) *$C \subseteq A$ and $B \subseteq D$, or*
(iv) *$B = C = G$ and $A = D$ and $E(A) = \emptyset$.*

Proof. Since we may assume that (ii) is false, it follows from the second axiom that $(A \cup C, B \cap D)$ has λ-order at most that of (A, B), and we may assume that equality holds, for otherwise (i) holds. Thus, by the first axiom, $(A \cup C, B \cap D) = (A, B)$ or (B, A). If $(A \cup C, B \cap D) = (A, B)$ then $C \subseteq A$ and $B \subseteq D$ and (iii) holds, and so we may assume that $(A \cup C, B \cap D) = (B, A)$. Hence $A \cup C = B$ and $B \cap D = A$. In particular, $A \subseteq B$, and since $A \cup B = G$, it follows that $B = G$, and $A = D$ since $B \cap D = A$.

By the second axiom applied to (D, C), (B, A), we deduce that either $(B \cup D, A \cap C)$ has λ-order at most that of (D, C) or $(B \cap D, A \cup C)$ has λ-order less than (B, A). In the second case, (i) holds, and if strict inequality holds in the first case, then (ii) holds. Thus we may assume that $(B \cup D, A \cap C)$ has the same λ-order as (D, C), and so $(B \cup D, A \cap C) = (D, C)$ or (C, D), by the first axiom. In the first case, $B \subseteq D$ and $C \subseteq A$, and (iii) holds, and so we may assume that $(B \cup D, A \cap C) = (C, D)$; that is, $C = G$ and $A = D$. Since $B = G$, it follows that (iv) holds. ∎

Given a tie-breaker λ, a separation (A, B) of G is *λ-robust* if for every separation (C, D) of A, one of $(C, B \cup D)$, $(D, B \cup C)$ has λ-order at least the λ-order of (A, B). Clearly a λ-robust separation is robust. The separation (A, B) is *doubly λ-robust* if both (A, B) and (B, A) are λ-robust.

(9.4) *Let (A, B), (C, D) be doubly λ-robust separations of G. Then (A, B) and (C, D) do not cross.*

Proof. By the symmetry, we may assume that of the four separations $(A \cap C, B \cup D)$, $(A \cap D, B \cup C)$, $(B \cap C, A \cup D)$, $(B \cap D, A \cup C)$, the first has smallest λ-order. Since $(C \cap A, D \cap A)$ is a separation of A and (A, B) is λ-robust, one of

$$(C \cap A, (D \cap A) \cup B), \qquad (D \cap A, (C \cap A) \cup B)$$

has λ-order at least that of (A, B). These separations are $(A \cap C, B \cup D)$ and $(A \cap D, B \cup C)$, respectively, and so, in view of the assumption in the first sentence of this proof, $(A \cap D, B \cup C)$ has λ-order at least that of (A, B). Similarly, $(B \cap C, A \cup D)$ has λ-order at least that of (C, D). By (9.3) applied to (B, A), (C, D), we deduce that either $C \subseteq B$ and $A \subseteq D$, or $A = C = G$ and $B = D$, and in either case (A, B), (C, D) do not cross. ∎

10. Tangle Tree-Decompositions

Let $\mathscr{T}_1$, $\mathscr{T}_2$ be tangles in a graph G. They are *indistinguishable* if one is a truncation of the other, that is, either $\mathscr{T}_1 \subseteq \mathscr{T}_2$ or $\mathscr{T}_2 \subseteq \mathscr{T}_1$, and otherwise they are *distinguishable*. A separation (A, B) of G *distinguishes $\mathscr{T}_1$ from $\mathscr{T}_2$* if $(A, B) \in \mathscr{T}_1$ and $(B, A) \in \mathscr{T}_2$.

(10.1) *Either there is a separation of G which distinguishes $\mathscr{T}_1$ from $\mathscr{T}_2$ or $\mathscr{T}_1$, $\mathscr{T}_2$ are indistinguishable and not both.*

Proof. Since there is a separation distinguishing $\mathscr{T}_1$ from $\mathscr{T}_2$ if and only if there is one distinguishing $\mathscr{T}_2$ from $\mathscr{T}_1$, we may assume that $\mathscr{T}_2$ has order at least that of $\mathscr{T}_1$. Then

$\mathscr{T}_1$ and $\mathscr{T}_2$ are distinguishable

$\Leftrightarrow \mathscr{T}_1 \nsubseteq \mathscr{T}_2$

$\Leftrightarrow$ there exists $(A, B) \in \mathscr{T}_1$ with $(A, B) \notin \mathscr{T}_2$

$\Leftrightarrow$ there exists $(A, B) \in \mathscr{T}_1$ with $(B, A) \in \mathscr{T}_2$

$\Leftrightarrow$ there is a separation distinguishing $\mathscr{T}_1$ from $\mathscr{T}_2$,

as required. ∎

Given a tie-breaker λ, a separation (A, B) which distinguishes $\mathscr{T}_1$ from $\mathscr{T}_2$ is a $(\mathscr{T}_1, \mathscr{T}_2)$-*distinction* if it has minimum λ-order of all separations which distinguish $\mathscr{T}_1$ from $\mathscr{T}_2$. From the first tie-breaker axiom, (A, B) is unique, and we may speak of *the* $(\mathscr{T}_1, \mathscr{T}_2)$-distinction. (Of course, different choices of the tie-breaker λ result in different $(\mathscr{T}_1, \mathscr{T}_2)$-distinctions in general.) There is a $(\mathscr{T}_1, \mathscr{T}_2)$-distinction if and only if $\mathscr{T}_1, \mathscr{T}_2$ are distinguishable.

(10.2) *If $\mathscr{T}_1, \mathscr{T}_2$ are distinguishable tangles in G, the $(\mathscr{T}_1, \mathscr{T}_2)$-distinction is doubly λ-robust.*

Proof. Let (A, B) be the $(\mathscr{T}_1, \mathscr{T}_2)$-distinction. Since (B, A) is the $(\mathscr{T}_2, \mathscr{T}_1)$-distinction, it suffices to show that (A, B) is λ-robust. Let (C, D) be a separation of A, and suppose that both $(C, B \cup D)$ and $(D, B \cup C)$ have λ-order strictly smaller than that of (A, B). Then $(C, B \cup D)$, $(D, B \cup C)$ have order at most that of (A, B) and hence less than the orders of $\mathscr{T}_1$ and $\mathscr{T}_2$. Since $(A, B) \in \mathscr{T}_1$ it follows that $(C, B \cup D) \in \mathscr{T}_1$ and $(D, B \cup C) \in \mathscr{T}_1$. Since (A, B) is the $(\mathscr{T}_1, \mathscr{T}_2)$-distinction it follows that $(B \cup D, C) \notin \mathscr{T}_2$ and $(B \cup C, D) \notin \mathscr{T}_2$, and hence $(C, B \cup D)$, $(D, B \cup C) \in \mathscr{T}_2$. But $(B, A) \in \mathscr{T}_2$, and $B \cup C \cup D = G$, contrary to the second tangle axiom. Thus one of $(C, B \cup D)$, $(D, B \cup C)$ has λ-order at least that of (A, B), and hence (A, B) is λ-robust, as required. ∎

(10.3) *Let $\mathscr{T}_1, ..., \mathscr{T}_n$ be mutually distinguishable tangles in a hypergraph G with $n \geq 1$, and let λ be a tie-breaker. Then there is a tree-decomposition (T, τ) of G, where $V(T) = \{t_1, ..., t_n\}$, with the following properties:*

(i) *For every $e \in E(T)$ and for $1 \leqslant i \leqslant n$, if T_1, T_2 are the components of $T \backslash e$ and $t_i \in V(T_1)$ then*

$$\left(\bigcup_{t \in V(T_1)} \tau(t), \bigcup_{t \in V(T_2)} \tau(t) \right) \notin \mathscr{T}_i.$$

(ii) *For all $i \neq j$ with $1 \leqslant i, j \leqslant n$, let e be the edge of the path of T between t_i and t_j making separations of smallest λ-order; then these separations are the $(\mathscr{T}_i, \mathscr{T}_j)$- and $(\mathscr{T}_j, \mathscr{T}_i)$-distinctions.*

Proof. For $i \neq j$ with $1 \leqslant i, j \leqslant n$, there is a $(\mathscr{T}_i, \mathscr{T}_j)$-distinction. Each of these separations is doubly λ-robust by (10.2), and so by (9.4) no two of them cross. By (9.1) there is a tree-decomposition (T, τ) of G such that

(i) for $1 \leqslant i, j \leqslant n$ with $i \neq j$, a unique edge of T makes the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction

(ii) for every $e \in E(T)$, there exist $i \neq j$ with $1 \leqslant i, j \leqslant n$ such that e makes the $(\mathscr{T}_i, \mathscr{T}_j)$- and $(\mathscr{T}_j, \mathscr{T}_i)$-distinctions.

For $1 \leqslant i \leqslant n$, we say $t_0 \in V(T)$ is a *home* for $\mathscr{T}_i$ if for every $e \in E(T)$,

$$\left(\bigcup_{t \in V(T_1)} \tau(t), \bigcup_{t \in V(T_2)} \tau(t)\right) \notin \mathscr{T}_i,$$

where T_1, T_2 are the components of $T \backslash e$ and $t_0 \in V(T_1)$.

(1) *For $t_0 \in T$ and $1 \leqslant i < j \leqslant n$, t_0 is not a home for both $\mathscr{T}_i$ and $\mathscr{T}_j$.*

For let e be an edge of T making the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction. Let T_1, T_2 be the components of $T \backslash e$, where the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction (A, B) is

$$\left(\bigcup_{t \in V(T_1)} \tau(t), \bigcup_{t \in V(T_2)} \tau(t)\right).$$

Then $(A, B) \in \mathscr{T}_i$ and $(B, A) \in \mathscr{T}_j$, and so if t_0 is a home for $\mathscr{T}_i$ then $t_0 \notin V(T_1)$, and if t_0 is a home for $\mathscr{T}_j$ then $t_0 \notin V(T_2)$. Since $t_0 \in V(T_1 \cup T_2)$, t_0 is not a home for both $\mathscr{T}_i$ and $\mathscr{T}_j$, as required.

For the moment, fix i with $1 \leqslant i \leqslant n$. An edge $e \in E(T)$ is *i-relevant* if the separations made by e have order less than the order of $\mathscr{T}_i$. Let us direct each i-relevant edge e so that

$$\left(\bigcup_{t \in V(T_1)} \tau(t), \bigcup_{t \in V(T_2)} \tau(t)\right) \in \mathscr{T}_i,$$

where T_1, T_2 are the components of $T \backslash e$ and $V(T_2)$ contains the head of e. We observe that

(2) *$t_0 \in V(T)$ is a home for $\mathscr{T}_i$ if and only if every i-relevant edge of T is directed towards t_0.*

Let H_i be the set of homes for $\mathscr{T}_i$.

(3) *$H_i \neq \varnothing$ and H_i is the set of vertices of a subtree of T.*

The second assertion follows from the first and (2). To show that $H_i \neq \varnothing$, it suffices (by an elementary property of trees) to show that for all i-relevant edges e, e' of T, if T_1, T_2 are the components of $T \backslash e$ with the head of e in $V(T_2)$, and T'_1, T'_2 are defined similarly, then $V(T_2) \cap V(T'_2) \neq \varnothing$. Now

$$\left(\bigcup_{t \in V(T_1)} \tau(t), \bigcup_{t \in V(T_2)} \tau(t)\right) \in \mathscr{T}_i$$

and

$$\left(\bigcup_{t \in V(T'_1)} \tau(t), \bigcup_{t \in V(T'_2)} \tau(t)\right) \in \mathscr{T}_i,$$

and so $T'_2 \not\subseteq T_1$ by the second tangle axiom; thus, $T_2 \cap T'_2$ is non-null, as required.

(4) *If $e \in E(T)$ has ends $x, y \in V(T)$, and $x \in H_i$, $y \notin H_i$, then e is i-relevant.*

For since $x \in H_i$ and $y \notin H_i$, some edge of T is directed towards x and not towards y. The only possible such edge is e, and so e is directed and hence i-relevant.

(5) *For $1 \leqslant i, j \leqslant n$, and $e \in E(T)$, e makes a separation which distinguishes $\mathscr{T}_i$ from $\mathscr{T}_j$ if and only if e lies on the (unique) minimal path of T between $V(H_i)$ and $V(H_j)$ and is i- and j-relevant.*

For if e makes a separation which distinguishes $\mathscr{T}_i$ from $\mathscr{T}_j$, this separation has order less than the smaller of the orders of $\mathscr{T}_i, \mathscr{T}_j$, and so e is i-relevant and j-relevant, and from (2), e lies on the unique minimal $H_i - H_j$ path in T. Conversely, if e lies on this path and is i- and j-relevant, then it makes a separation (A, B) with $(A, B) \in \mathscr{T}_i$ and $(B, A) \in \mathscr{T}_j$, by definition of H_i and H_j, as required.

(6) *For $1 \leqslant i \leqslant n$, $|H_i| = 1$.*

For suppose that $|H_i| \geqslant 2$ for some i. Choose $t_1, t_2 \in H_i$, distinct and adjacent in T (this is possible by (3)) joined by an edge e. Then e is not i-relevant. Choose j, k with $j \neq k$ and $1 \leqslant j, k \leqslant n$ such that e makes the $(\mathscr{T}_j, \mathscr{T}_k)$-distinction. Let P be the minimal $H_j - H_k$ path in T. Then $e \in E(P)$ by (5), and so $j, k \neq i$. Let $f \in E(T)$ make the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction. Then by (5), $f \in E(P)$. Since f is i-relevant and e is not, f makes a separation of order (and hence λ-order) strictly smaller than that of the $(\mathscr{T}_j, \mathscr{T}_k)$-distinction, and by (5) makes a separation of that order which distinguishes $\mathscr{T}_j$ from $\mathscr{T}_k$, a contradiction, as required.

(7) *$H_1 \cup \cdots \cup H_n = V(T)$.*

For suppose that $t_0 \in V(T) - (H_1 \cup \cdots \cup H_n)$. Since $n \neq 0$, $|V(T)| \geqslant 2$, and so there is a neighbour of t_0 in T. Let the edges of T incident with t_0 be $e_1, ..., e_k$, let T_p be the component of $T \backslash e_p$ not containing t_0, and let T'_p be the other component of $T \backslash e_p$ $(1 \leqslant p \leqslant k)$. The separations made by $e_1, ..., e_k$ are all distinct, since each of them is the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction for some i, j, and the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction is made by a unique edge, from our initial choice of the tree-decomposition. Thus we may assume, by the first tie-breaker axiom, that the separations made by e_1 have λ-order strictly more than the separations made by $e_2, ..., e_k$. Choose i, j with $i \neq j$ such that

$$\left(\bigcup_{t \in V(T_1)} \tau(t), \bigcup_{t \in V(T'_1)} \tau(t) \right)$$

is the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction. Let P be the minimal $H_i - H_j$ path in T. Then $e_1 \in E(P)$, and since $t_0 \notin H_i \cup H_j$, $E(P)$ contains one of $e_2, ..., e_k$, say e_2. Now

$$\left(\bigcup_{t \in V(T_2)} \tau(t), \bigcup_{t \in V(T_2')} \tau(t) \right)$$

has λ-order strictly less than that of the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction and hence has order at most that of the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction. By (5), e_2 makes a separation which distinguishes $\mathscr{T}_i$ from $\mathscr{T}_j$, with λ-order strictly smaller than that of the $(\mathscr{T}_i, \mathscr{T}_j)$-distinction, a contradiction.

Let $H_i = \{t_i\}$ $(1 \leqslant i \leqslant n)$; then the theorem is satisfied. ∎

We call the tree-decomposition of (10.3) the *standard* tree-decomposition of G relative to $\mathscr{T}_1, ..., \mathscr{T}_n$.

From (10.3) we deduce a corollary mentioned earlier. We merely sketch the proof since we do not need the result.

(10.4) *In any hypergraph G there are at most $|V(G)|$ maximal tangles.*

Proof. Let $\mathscr{T}_1, ..., \mathscr{T}_n$ be the distinct maximal tangles in G, and let λ be a tie-breaker. Since they are mutually distinguishable, there is a standard tree-decomposition (T, τ). Let $e, f \in E(T)$ be distinct, making separations (A, B) and (C, D), say, where $A \subseteq C$ and $D \subseteq B$. If $V(A) = V(C)$ then it follows easily that $A = C$, $B = D$, a contradiction; thus $V(A) \subset V(C)$ and similarly $V(D) \subset V(B)$. From this one can show that $|E(T)| \leqslant |V(G)| - 1$, and hence $n = |V(T)| \leqslant |V(G)|$, as required. ∎

11. Structure Relative to a Tangle

Now we come to the last main result of the paper. We have seen in (5.2) that if G has small tangle number, then it has a tree-decomposition of small width. Our problem here is, suppose that G has large tangle number, but relative to each high order tangle the graph has a structure or decomposition of a certain kind X, say; what can we infer about the global structure of G from this local knowledge? One might guess that G should have a tree-decomposition into pieces each with structure X, but that is false. Nevertheless, it turns out that G has a tree-decomposition into pieces which "almost" have structure X, and we need to know this for an application in [6].

A *design* is a pair (H, M), where H is a hypergraph and M is a set of subsets of $V(H)$. If (T, τ) is a tree-decomposition of a hypergraph G and $t_0 \in V(T)$, and t_0 has neighbours $t_1, \ldots, t_k$ in T, then

$$(\tau(t_0), \{V(\tau(t_0) \cap \tau(t_i)) : 1 \leqslant i \leqslant k\})$$

is a design, called the *design* of t_0 *in* (T, τ). If $\mathscr{S}$ is a class of designs, a tree-decomposition (T, τ) is said to be *over* $\mathscr{S}$ if for each $t_0 \in V(T)$, $\mathscr{T}$ contains the design of t_0 in (T, τ).

Let (H, M), (H', M') be designs and let $Z \subseteq V(H')$ be such that

(i) H is a subhypergraph of H' and $V(H') - V(H) \subseteq Z$

(ii) every edge of H' is an edge of H

(iii) for every $X \in M'$ with $X \neq Z$, $X \cap V(H) \in M$.

(Z may or may not be a member of M'.) In these circumstances, we say that (H', M') is an *n-enlargement of* (H, M) for every integer $n \geqslant |Z|$. If $\mathscr{S}$ is a class of designs, we denote the class of all n-enlargements of members of $\mathscr{S}$ by $\mathscr{S}^n$. For any integer $n \geqslant 0$, we denote by $\mathscr{R}_n$ the class of all designs (H, M) with $|V(H)| \leqslant n$.

A *location* in a hypergraph G is a set $\{(A_1, B_1), \ldots, (A_k, B_k)\}$ of separations of G such that $A_i \subseteq B_j$ for all distinct i, j with $1 \leqslant i, j \leqslant k$. If $\{(A_1, B_1), \ldots, (A_k, B_k)\}$ is a location in G, then

$$(G \cap B_1 \cap \cdots \cap B_k; \{V(A_i \cap B_i) : 1 \leqslant i \leqslant k\})$$

is a design, which we call the *design of the location.*

Let $\theta \geqslant 1$ be an integer, and let $\mathscr{S}$ be a class of designs. We say that $\mathscr{S}$ is *θ-pervasive* in a hypergraph G if for every subhypergraph G' of G and every tangle $\mathscr{T}$ in G' of order $\geqslant \theta$ there is a location $\mathscr{L}$ in G' such that $\mathscr{L} \subseteq \mathscr{T}$ and the design of $\mathscr{L}$ belongs to $\mathscr{S}$. Our object is to deduce information about the global structure of G from the knowledge that a certain class of designs is θ-pervasive. We show

(11.1) *For any $\theta \geqslant 1$, let $\mathscr{S}$ be a class of designs which is θ-pervasive in a hypergraph G; then G has a tree-decomposition over $\mathscr{S}^{3\theta-2} \cup \mathscr{R}_{4\theta-3}$.*

We need the following lemma.

(11.2) *Let $\theta \geqslant 1$, let $\mathscr{S}$ be θ-pervasive in G, and let $Z \subseteq V(G)$ with $|Z| = 3\theta - 2$. Then either*

(i) *there is a separation (A, B) of G of order $< \theta$ with*

$$|(Z \cup V(A)) \cap V(B)|, |(Z \cup V(B)) \cap V(A)| \leqslant 3\theta - 3$$

or

(ii) *there is a location* $\{(A_1, B_1), ..., (A_k, B_k)\}$ *in* G, *with design in* $\mathscr{S}$, *such that for* $1 \leqslant i \leqslant k$,

$$|Z \cap V(A_i)| \leqslant |V(A_i \cap B_i)| < \theta.$$

Proof. Let $\mathscr{T}$ be the set of all separations (A, B) of G of order $<\theta$ such that $|Z \cap V(A)| \leqslant |V(A \cap B)|$. Since $|Z| > 3(\theta - 1)$ the second and third tangle axioms hold for $\mathscr{T}$. Suppose the first does not; then there is a separation (A, B) of order $<\theta$ such that $|Z \cap V(A)|$, $|Z \cap V(B)| > |V(A \cap B)|$. But then

$$|(Z \cup V(A)) \cap V(B)| = |V(A \cap B)| + |Z - V(A)|$$
$$< |Z \cap V(A)| + |Z - V(A)| = |Z| = 3\theta - 2$$

and similarly $|(Z \cup V(B)) \cap V(A)| \leqslant 3\theta - 3$, and so (i) holds. We may assume then that $\mathscr{T}$ is a tangle of order θ.

Since $\mathscr{S}$ is θ-pervasive, there is a location $\{(A_1, B_1), ..., (A_k, B_k)\} \subseteq \mathscr{T}$ with design in $\mathscr{S}$. Thus for $1 \leqslant i \leqslant k$, $|Z \cap V(A_i)| \leqslant |V(A_i \cap B_i)| < \theta$, and so (ii) holds, as required. ∎

If (H, M) is a design and $Z \subseteq V(H)$ then $(H, M \cup \{Z\})$ is a design, which we call the *Z-extension* of (H, M). In order to prove our main result (11.1) it is convenient for inductive purposes to prove a somewhat strengthened form, the following ((11.1) may be derived from this by setting $Z = \varnothing$).

(11.3) *Let* $\mathscr{S}$ *be a class of designs, and let* $\theta \geqslant 1$. *Let* G *be a hypergraph such that* $\mathscr{S}$ *is* θ*-pervasive in* G, *and let* $Z \subseteq V(G)$ *with* $|Z| \leqslant 3\theta - 2$. *Then there is a tree-decomposition* (T, τ) *of* G *over* $\mathscr{S}^{3\theta-2} \cup \mathscr{R}_{4\theta-3}$, *such that for some* $t_0 \in V(T)$, $Z \subseteq V(\tau(t_0))$ *and* $\mathscr{S}^{3\theta-2} \cup \mathscr{R}_{4\theta-3}$ *contains the Z-extension of the design of* t_0 *in* (T, τ).

Proof. Let us remark, first, that from the definition of θ-pervasive, if $\mathscr{S}$ is θ-pervasive in G then it is θ-pervasive in every subhypergraph of G. Let $\mathscr{S}' = \mathscr{S}^{3\theta-2} \cup \mathscr{R}_{4\theta-3}$. For fixed $\mathscr{S}$, θ, we prove that the result holds for all G, Z by induction on $|V(G)|$. Thus, let us assume that it holds for all G', Z' with $|V(G')| < |V(G)|$. First we show that it holds for G, Z if $|Z| = 3\theta - 2$.

Therefore, let $|Z| = 3\theta - 2$. By (11.2), one of the following two cases applies.

Case 1. There is a separation (A_1, A_2) of G of order $<\theta$, with

$$|(Z \cup V(A_1)) \cap V(A_2)|, |(Z \cup V(A_2)) \cap V(A_1)| \leqslant 3\theta - 3.$$

Let $Z_1 = (Z \cup V(A_2)) \cap V(A_1)$, $Z_2 = (Z \cup V(A_1)) \cap V(A_2)$. Then for $i = 1, 2$, $Z_i \subseteq V(A_i)$ and $|Z_i| \leqslant 3\theta - 3$. Since $|Z_1| < |Z|$ and so $Z \nsubseteq Z_1$, it follows that $V(A_1) \neq V(G)$, and so the result holds for A_1, Z_1, and similarly for A_2, Z_2 by our inductive hypothesis. Since $\mathscr{S}$ is θ-pervasive in A_1 and in A_2, it follows that for $i = 1, 2$, there is a tree-decomposition (T_i, τ_i) of A_i over $\mathscr{S}'$, and there exists $t_i \in V(T_i)$ such that $Z_i \subseteq V(\tau_i(t_i))$ and $\mathscr{S}'$ contains the Z_i-extension of the design of t_i in (T_i, τ_i). We choose T_1, T_2 to be disjoint. Take a new vertex t_0, and let T be the tree with vertex set $V(T_1) \cup V(T_2) \cup \{t_0\}$, where $T \backslash t_0 = T_1 \cup T_2$ and t_0 is adjacent to t_1, t_2. Let $\tau(t_0)$ be the hypergraph with vertex set $Z \cup V(A_1 \cap A_2)$ and with no edges, and let $\tau(t) = \tau_i(t)$ if $t \in V(T_i)$ $(i = 1, 2)$. Then (T, τ) is a tree-decomposition of G, as is easily seen. The design of t_0 in (T, τ) is $(\tau(t_0), \{Z_1, Z_2\})$, which is in $\mathscr{R}_{4\theta-3}$, since

$$|V(\tau(t_0))| = |Z \cup V(A_1 \cap A_2)| \leqslant |Z| + |V(A_1 \cap A_2)| \leqslant (3\theta - 2) + (\theta - 1),$$

and the Z-extension of this design is also in $\mathscr{R}_{4\theta-3}$, for the same reason. For $i = 1, 2$ and each $t \in V(T_i)$, the design of t in (T, τ) equals the design of t in (T_i, τ_i) (or its Z_i-extension if $t = t_i$) and so belongs to $\mathscr{S}'$. Hence the theorem holds in this case.

Case 2. There is a location $\{(A_1, B_1), ..., (A_k, B_k)\}$ in G with design in $\mathscr{S}$, such that for $1 \leqslant i \leqslant k$,

$$|Z \cap V(A_i)| \leqslant |V(A_i \cap B_i)| < \theta.$$

For $1 \leqslant i \leqslant k$, let $Z_i = (Z \cup V(B_i)) \cap V(A_i)$. Then $|Z_i| \leqslant 2(\theta - 1) \leqslant 3\theta - 2$, and $Z_i \subseteq V(A_i)$. Also,

$$|Z \cap V(A_i)| < \theta \leqslant 3\theta - 2 = |Z \cap V(G)|,$$

and so $V(A_i) \neq V(G)$. By our inductive hypothesis, there is a tree-decomposition (T_i, τ_i) of A_i over $\mathscr{S}'$, and there exists $t_i \in V(T_i)$ such that $Z_i \subseteq V(\tau_i(t_i))$ and $\mathscr{S}'$ contains the Z_i-extension of the design of t_i in (T_i, τ_i). We choose $T_1, ..., T_k$ to be disjoint. Take a new vertex t_0, and let T be the tree with vertex set $V(T_1) \cup \cdots \cup V(T_k) \cup \{t_0\}$, where $T \backslash t_0 = T_1 \cup \cdots \cup T_k$ and t_0 is adjacent to $t_1, ..., t_k$. Let $\tau(t_0)$ be the hypergraph with vertex set

$$V(G \cap B_1 \cap B_2 \cap \cdots \cap B_k) \cup Z$$

and with edge set and incidence relation the same as those of $G \cap B_1 \cap B_2 \cap \cdots \cap B_k$. Let $\tau(t) = \tau_i(t)$ if $t \in V(T_i)$ $(1 \leqslant i \leqslant k)$. Then (T, τ) is a tree-decomposition of G, as is easily seen. Let us examine the designs of the vertices of T in (T, τ). First, let $1 \leqslant i \leqslant k$ and let $t \in V(T_i)$ with $t \neq t_i$.

Then the design of t in (T, τ) equals the design of t in (T_i, τ_i), and hence this design belongs to $\mathscr{S}'$. Secondly, let $1 \leqslant i \leqslant k$ and let $t = t_i$; the design of t in (T, τ) is the Z_i-extension of the design of t in (T_i, τ_i) and hence also belongs to $\mathscr{S}'$. Finally, the design of t_0 in (T, τ) is $(\tau(t_0), \{Z_i : 1 \leqslant i \leqslant k\})$ and its Z-extension is $(\tau(t_0), \{Z_i : 1 \leqslant i \leqslant k\} \cup \{Z\})$. But these designs are both $|Z|$-enlargements of

$$(G \cap B_1 \cap \cdots \cap B_k, \{V(A_i \cap B_i) : 1 \leqslant i \leqslant k\}) \in \mathscr{S},$$

and so they both belong to $\mathscr{S}^{3\theta - 2} \subseteq \mathscr{S}'$, as required.

Thus, we have proved that the result holds for G, Z when $|Z| = 3\theta - 2$. Now let $Z \subseteq V(G)$ with $|Z| \leqslant 3\theta - 2$. If $|V(G)| < 3\theta - 2$ then $(G, \{Z\}) \in \mathscr{R}_{3\theta - 3} \subseteq \mathscr{S}'$, and so the desired tree-decomposition (T, τ) exists with T a 1-vertex tree. We may assume then that $|V(G)| \geqslant 3\theta - 2$. Choose $Z' \subseteq V(G)$ with $Z \subseteq Z'$ and $|Z'| = 3\theta - 2$. As we have seen above, the result holds for G, Z', and so there is a tree-decomposition (T_1, τ_1) of G over $\mathscr{S}'$, such that for some $t_1 \in V(T_1)$, $Z' \subseteq V(\tau_1(t_1))$ and $\mathscr{S}'$ contains the Z'-extension of the design of t_1 in (T_1, τ_1). Take a new vertex t_0, and let T be the tree with vertex set $V(T_1) \cup \{t_0\}$, where $T \backslash t_0 = T_1$ and t_0 is adjacent to t_1. Let $\tau(t_0)$ be the hypergraph with vertex set Z' and no edges, and for $t \in V(T_1)$, let $\tau(t) = \tau_1(t)$. Then (T, τ) is a tree-decomposition of G. For $t \in V(T)$ with $t \neq t_0, t_1$, the design of t in (T, τ) equals the design of t in (T_1, τ_1) and hence belongs to $\mathscr{S}'$. The design of t_1 in (T, τ) is the Z'-extension of the design of t_1 in (T_1, τ_1) and hence belongs to $\mathscr{S}'$. Finally, the design of t_0 in (T, τ) is $(\tau(t_0), \{Z'\})$, and the Z-extension of this is $(\tau(t_0), \{Z, Z'\})$, and both of these belong to $\mathscr{R}_{3\theta - 2} \subseteq \mathscr{S}'$. This completes the proof. ▮

We remark that in essence (11.1) generalizes (5.2). For let $\mathscr{S} = \emptyset$. Then it follows from (11.1) that if G is a hypergraph with no tangle of order θ (and so $\mathscr{S}$ is θ-pervasive) then G has a tree-decomposition over $\mathscr{R}_{4\theta - 3}$, and hence $\omega(G) \leqslant 4\theta - 4$; in other words, $\omega(G) \leqslant 4\theta(G)$. Apart from the size of the multiplicative constant, this is the main part of (5.2).

12. Tangles and Matroids

Finally, let us discuss some matroidal aspects of tangles. Let $\mathscr{T}$ be a tangle in a hypergraph G, of order θ. For $X \subseteq V(G)$, let us define $r(X)$ to be the least order of a separation $(A, B) \in \mathscr{T}$ with $X \subseteq V(A)$, if one exists, and θ otherwise.

(12.1) *r is the rank function of a matroid on $V(G)$.*

Proof. We must check [8] that

(i) r is integral-valued

(ii) for $X \subseteq V(G)$, $0 \leqslant r(X) \leqslant |X|$

(iii) for $X \subseteq Y \subseteq V(G)$, $r(X) \leqslant r(Y)$

(iv) for $X, Y \subseteq V(G)$, $r(X \cup Y) + r(X \cap Y) \leqslant r(X) + r(Y)$.

(i) and (iii) are clear. For (ii), certainly $r(X) \geqslant 0$. Since $r(X) \leqslant \theta$, we may assume that $|X| < \theta$. Let K be the hypergraph with $V(K) = X$, $E(K) = \emptyset$. Since $(G, K) \notin \mathscr{T}$ and has order $< \theta$, it follows that $(K, G) \in \mathscr{T}$, and so

$$r(X) \leqslant |V(K \cap G)| \leqslant |X|.$$

This verifies (ii). For (iv), let $X, Y \subseteq V(G)$. Since $r(X \cap Y) \leqslant r(Y)$ and $r(X \cup Y) \leqslant \theta$, we may assume that $r(X) < \theta$ and similarly $r(Y) < \theta$. Choose $(A, B) \in \mathscr{T}$ of order $r(X)$ with $X \subseteq V(A)$ and $(C, D) \in \mathscr{T}$ of order $r(Y)$ with $Y \subseteq V(C)$. We claim that $r(X \cap Y)$ is at most the order of $(A \cap C, B \cup D)$; for this is true if $(A \cap C, B \cup D)$ has order $\geqslant \theta$, and otherwise $(A \cap C, B \cup D) \in \mathscr{T}$, and the claim follows since $X \cap Y \subseteq V(A \cap C)$. Similarly, $r(X \cup Y)$ is at most the order of $(A \cup C, B \cap D)$, by (2.2). Since the sum of the orders of (A, B) and (C, D) equals the sum of the order of $(A \cap C, B \cup D)$ and $(A \cup C, B \cap D)$, the result follows. ∎

Thus, given $\mathscr{T}$, G as before, let us say that $X \subseteq V(G)$ is *free* if $|X| \leqslant \theta$ and there is no $(A, B) \in \mathscr{T}$ of order $< |X|$ with $X \subseteq V(A)$. From (12.1) we deduce

(12.2) *The free sets are the independent sets of a matroid on* $V(G)$ *with rank function* r *as in* (12.1).

We shall need (12.2) in a later paper. Incidentally, we do not know which matroids can arise this way, but they are not just the gammoids [8].

Secondly, for the matroid theorist it is a little unnatural to define the order of a separation (A, B) of a graph to be $|V(A \cap B)|$, as we have done. From the viewpoint of matroid theory, a more significant number is the *Tutte-order*, defined to be

$$|V(A \cap B)| + 1 + \kappa(G) - \kappa(A) - \kappa(B),$$

where $\kappa(F)$ denotes the number of components of F, for a subgraph F of G; for the Tutte-order of a separation (A, B) equals

$$r(E(A)) + r(E(B)) - r(E(G)) + 1,$$

where r is the rank function of the polygon matroid of G. One can define both "Tutte-tangles" and "Tutte-branch-width" using Tutte-order instead of order, and the analogue of (4.3) holds. Indeed, this definition of the order of a separation extends to general matroids in the natural way, and again the analogue of (4.3) holds (with essentially the same proof). We suspect, but have not shown, that in a graph, Tutte-tangles and tangles are essentially the same objects. Some evidence for this lies in the fact that, for a connected planar graph, there is a 1–1 correspondence between its tangles and the tangles in a geometric dual [5].

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