Trees

Trees, Binary Search Trees, Heaps & Applications

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Abstract

These are lecture notes used in CSCE 156 (Computer Science II), CSCE 235 (Discrete Structures) and CSCE 310 (Data Structures & Algorithms) at the University of Nebraska—Lincoln.

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1 Introduction

Motivation: we want a data structure to store elements that offers efficient, arbitrary retrieval (search), insertion, and deletion.

- Array-based Lists
  - $O(n)$ insertion and deletion
  - Fast index-based retrieval
  - Efficient binary search if sorted
- Linked Lists
  - Efficient, $O(1)$ insert/delete for head/tail

  - Inefficient, $O(n)$ arbitrary search/insert/delete
  - Efficient binary search not possible without random access

- Stacks and queues are efficient, but are restricted access data structures
- Possible alternative: Trees
- Trees have the potential to provide $O(\log n)$ efficiency for all operations

2 Definitions & Terminology

- A *tree* is an acyclic graph
- For our purposes: a *tree* is a collection of *nodes* (that can hold keys, data, etc.) that are connected by *edges*
- Trees are also *oriented*: each node has a parent and children
- A node with no parents is the *root* of the tree, all child nodes are oriented downward
- Nodes not immediately connected can have an ancestor, descendant or cousin relationship
- A node with no children is a *leaf*
- A tree such that all nodes have at most two children is called a *binary tree*
- A binary tree is also oriented horizontally: each node may have a left and/or a right child
- Example: see Figure 1
- A *path* in a tree is a sequence nodes connected by edges
- The *length* of a path in a tree is the number of edges in the path (which equals the number of nodes in the path minus one)
- A path is *simple* if it does not traverse nodes more than once (this is the default type of path)
- The *depth* of a node u is the length of the (unique) path from the root to u
- The depth of the root is 0
- The depth of a tree is the maximal depth of any node in the tree (sometimes the term *height* is used)

- All nodes of the same depth are considered to be at the same *level*
- A binary tree is *complete* (also called *full* or *perfect*) if all nodes are present at all levels 0 up to its depth d
- A sub-tree rooted at a node u is the tree consisting of all descendants with u oriented as the root

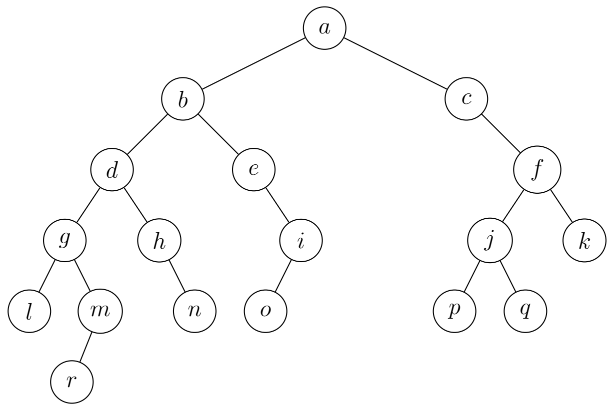


Figure 1: A Binary Tree

Properties:

- In a tree, all nodes are connected by exactly one unique path
- The maximum number of nodes at any level k is 2^k
- Thus, the maximum number of nodes n for any binary tree of depth d is:

$$n = 2^0 + 2^1 + 2^2 + \cdots + 2^{d-1} + 2^d = \sum_{k=0}^{d} 2^k = 2^{d+1} - 1$$

- Given a *full binary tree* with n nodes in it has depth:

$$d = \log(n+1) - 1$$

- That is, $d = O(\log n)$

Motivation: if we can create a tree-based data structure with operations proportional to its depth, then we could potentially have a data structure that allows retrieval/search, insertion, and deletion in $O(\log n)$-time.

3 Tree Traversal

- Given a tree, we need a way to enumerate elements in a tree
- Many algorithms exist to iterate over the elements in a tree
- We'll look at several variations on a depth-first-search

3.1 Preorder Traversal

- A preorder traversal strategy visits nodes in the following order: root; left-sub-tree; right-sub-tree
- An example traversal on the tree in Figure 1:

$$a, b, d, g, l, m, r, h, n, e, i, o, c, f, j, p, q, k$$

- Applications:
  - Building a tree, traversing a tree, copying a tree, etc.
  - Efficient stack-based implementation
  - Used in prefix notation (polish notation); used in languages such as Lisp/Scheme

3.2 Inorder Traversal

- An inorder traversal strategy visits nodes in the following order: left-sub-tree; root; right-sub-tree
- An example traversal on the tree in Figure 1:

$$l, g, r, m, d, h, n, b, e, o, i, a, c, p, j, q, f, k$$

- Applications:
  - Enumerating elements in order in a binary search tree
  - Expression trees

3.3 Postorder Traversal

- A postorder traversal strategy visits nodes in the following order: left-sub-tree; right-sub-tree; root

- An example traversal on the tree in Figure 1:

$$l, r, m, g, n, h, d, o, i, e, b, p, q, j, k, f, c, a$$

- Applications:
  - Topological sorting
  - Destroying a tree when manual memory management is necessary (roots are the last thing that get cleaned up)
  - Reverse polish notation (operand-operand-operator, unambiguous, used in old HP calculators)
  - PostScript (Page Description Language)

3.4 Breadth-First Search Traversal

- Breadth-First Search (BFS) traversal is a general graph traversal strategy that explores local or close nodes first before traversing "deeper" into the graph
- When applied to an oriented binary tree, BFS explores the tree level-by-level (top-to-bottom, left-to-right)

3.5 Implementations & Data Structures

- Reference based implementation: `TreeNode<T>`
  - Owns (through composition) references to: `leftChild`, `rightChild`, `parent`
  - Can use either *sentinel* nodes or `null` to indicate missing children and parent
- `BinaryTree<T>` owns a `root`
- SVN examples: `unl.cse.bst`

3.5.1 Preorder Implementations

Input : A binary tree node u
Output: A preorder traversal of the nodes in the subtree rooted at u

```
print u
preOrderTraversal(u → leftChild)
preOrderTraversal(u → rightChild)
```

Algorithm 1: Recursive Preorder Tree Traversal

Stack-based implementation:

- Initially, we push the tree's root into the stack
- Within a loop, we pop the top of the stack and process it
- We need to push the node's children for future processing
- Since a stack is LIFO, we push the right child first.

INPUT : A binary tree, T
OUTPUT: A preorder traversal of the nodes in T

$S \leftarrow$ empty stack
`push` T's root onto S
WHILE S *is not empty* DO
    node $\leftarrow$ `S.pop`
    `push` *node*'s right-child onto S
    `push` *node*'s left-child onto S
    process *node*
END

Algorithm 2: Stack-based Preorder Tree Traversal

3.5.2 Inorder Implementation

Stack-based implementation:

- The same basic idea: push nodes onto the stack as you visit them
- However, we want to delay processing the node until we've explored the left-sub-tree
- We need a way to tell if we are visiting the node for the first time or returning from the left-tree exploration
- To achieve this, we allow the node to be null
- If null, then we are returning from a left-tree exploration, pop the top of the stack and process (then push the right child)
- If not null, then we push it to the stack for later processing, explore the left child

```
Input   : A binary tree, T
Output: An inorder traversal of the nodes in T
S ← empty stack
u ← root
while S is not empty Or u ≠ null do
    if u ≠ null then
        push u onto S
        u ← u.leftChild
    else
        u ← S.pop
        process u
        u ← u.rightChild
    end
end
```

Algorithm 3: Stack-based Inorder Tree Traversal

3.5.3 Postorder Implementation

Stack-based implementation:

- Same basic ideas, except that we need to distinguish if we're visiting the node for the first time, second time or last (so that we can process it)
- To achieve this, we keep track of *where* we came from: a parent, left, or right node
- We keep track of a previous and a current node

```
INPUT  : A binary tree, T
OUTPUT: A postorder traversal of the nodes in T
S ← empty stack
prev ← null
push root onto S
WHILE S is not empty DO
    curr ← S.peek
    IF prev = null Or prev.leftChild = curr Or prev.rightChild = curr THEN
        IF curr.leftChild ≠ null THEN
            push curr.leftChild onto S
        ELSE IF curr.rightChild ≠ null THEN
            push curr.rightChild onto S
        END
    ELSE IF curr.leftChild = prev THEN
        IF curr.rightChild ≠ null THEN
            push curr.rightChild onto S
        END
    ELSE
        process curr
        S.pop
    END
    prev ← curr
END
```

Algorithm 4: Stack-based Postorder Tree Traversal

3.5.4 BFS Implementation

```
INPUT   : A binary tree, T
OUTPUT: A BFS traversal of the nodes in T
Q ← empty queue
enqueue T's root into Q
WHILE Q is not empty DO
    node ← Q.dequeue
    enqueue node's left-child onto Q
    enqueue node's right-child onto Q
    print node
END
```

Algorithm 5: Queue-based BFS Tree Traversal

3.5.5 Tree Walk Implementations

- Simple rules based on local information: where you are and where you came from
- No additional data structures required
- Traversal is a "walk" around the perimeter of the tree
- Can use similar rules to determine when the current node should be processed to achieve pre, in, and postorder traversals
- Need to take care with corner cases (when current node is the root or children are missing)
- Pseudocode presented Algorithm 6

3.6 Operations

Basic Operations:

- Search for a particular element/key
- Adding an element
  - Add at most shallow available spot
  - Add at a random leaf

```
Input  : A binary tree, T
Output: A Tree Walk around T
curr ← root
prevType ← parent
while curr ≠ null do
    if prevType = parent then
        //preorder: process curr here
        if curr.leftChild exists then
            //Go to the left child:
            curr ← curr.leftChild
            prevType ← parent
        else
            curr ← curr
            prevType ← left
        end
    else if prevType = left then
        //inorder: process curr here
        if curr.rightChild exists then
            //Go to the right child:
            curr ← curr.rightChild
            prevType ← parent
        else
            curr ← curr
            prevType ← right
        end
    else if prevType = right then
        //postorder: process curr here
        if curr.parent = null then
            //root has no parent, we're done traversing
            curr ← curr.parent
        //are we at the parent's left or right child?
        else if curr = curr.parent.leftChild then
            curr ← curr.parent
            prevType ← left
        else
            curr ← curr.parent
            prevType ← right
        end
    end
end
```

Algorithm 6: Tree Walk based Tree Traversal

  - Add internally, shift nodes down by some criteria

- Removing elements

  - Removing leaves
  - Removing elements with one child
  - Removing elements with two children

Other Operations:

- Compute the total number of nodes in a tree
- Compute the total number of leaves in a tree
- Given an item or node, compute its depth
- Compute the depth of a tree

4 Binary Search Trees

Regular binary search trees have little structure to their elements; search, insert, delete operations are still linear with respect to the number of tree nodes, $O(n)$. We want a data structure with operations proportional to its depth, $O(d)$. To this end, we add structure and order to tree nodes.

- Each node has an associated *key*
- Binary Search Tree Property: For every node u with key u_k in T

  1. Every node in the left-sub-tree of u has keys *less* than u_k
  2. Every node in the right-sub-tree of u has keys *greater* than u_k

- Duplicate keys can be handled, but you must be consistent and not guaranteed to be contiguous
- Alternatively: do not allow duplicate keys or define a key scheme that ensures a *total order*
- Inductive property: all sub-trees are also binary search trees
- A full example can be found in Figure 2

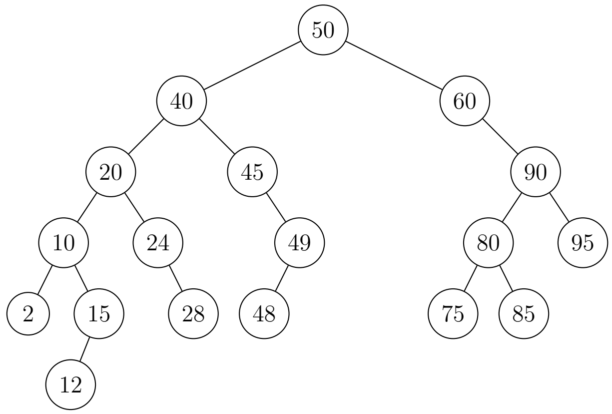


Figure 2: A Binary Search Tree

4.1 Basic Operations

Observation: a binary search tree has more *structure*: the key in each node provides information on where a node is *not* located. We will exploit this structure to achieve $O(\log n)$ operations.

Search/retrieve

- Goal: find a node (and its data) that matches a given key k
- Start at the node
- At each node u, compare k to u's key, u_k:
  - If equal, element found, stop and return
  - If $k < u_k$, traverse to u's left-child
  - If $k > u_k$, traverse to u's right-child
- Traverse until the sub-tree is empty (element not found)
- Analysis: number of comparisons is bounded by the depth of the tree, $O(d)$

INPUT : A binary search tree, T, a key k

OUTPUT: The tree node $u \in T$ whose key, u_k matches k

$u \leftarrow T$'s root
WHILE $u \neq \phi$ DO
    IF $u_k = k$ THEN
        output u
    END
    ELSE IF $u_k > k$ THEN
        $u \leftarrow u$'s left-child
    ELSE IF $u_k < k$ THEN
        $u \leftarrow u$'s left-child
END
output ϕ

Algorithm 7: Search algorithm for a binary search tree

Insert

- Insert new nodes as leaves
- To determine where it should be inserted: traverse the tree as above
- Insert at the first available spot (first missing child node)
- Analysis: finding the available location is $O(d)$, inserting is just reference juggling, $O(1)$

Delete

- Need to first *find* the node u to delete, traverse as above, $O(d)$
- If u is a leaf (no children): its safe to simply delete it
- If u has one child, then we can "promote" it to u's spot (u's parent will now point to u's child)
- If u has two children, we need to find a way to preserve the BST property
  - Want to minimally change the tree's structure
  - Need the operation to be efficient
  - Find the minimum element of the greater nodes (right sub-tree) or the maximal element of the lesser nodes (left sub-tree)
  - Such an element will have at most one child (which we know how to delete)

- Delete it and store off the key/data
- Replace *us* key/data with the contents of the minimum/maximum element

- Analysis:
  - Search/Find: $O(d)$
  - Finding the min/max: $O(d)$
  - Swapping: $O(1)$
  - In total: $O(d)$
- Examples illustrated in Figure 3

5 Balanced Binary Search Trees

Motivation:

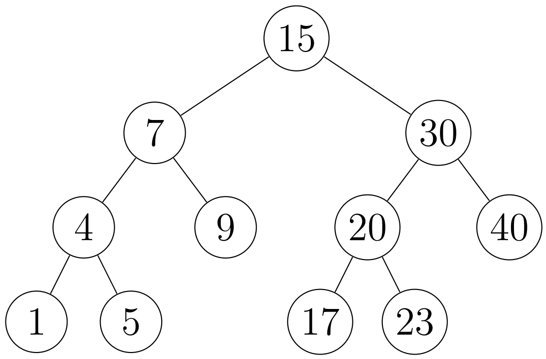
- Ideally, a full binary tree has depth $d = O(\log n)$
- General BSTs can degenerate such that $d = O(n)$
- Need a way to preserve the BST property while at the same time limiting the depth to $O(\log n)$
- Solution: Balanced Binary Search Trees
- BSTs that efficiently reorganize themselves such that the BST Property is preserved and that the depth is restricted to $O(\log n)$
- Some types of Balanced BSTs:
  - B-trees (and 2-3 trees)
  - AVL Trees
  - Red-Black Trees
  - Splay trees

5.1 2-3 Trees

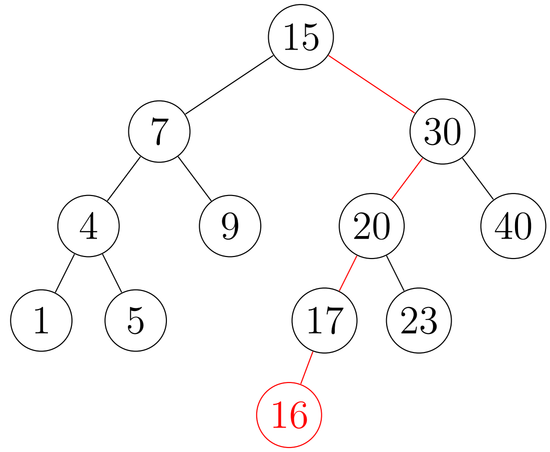
See 310 Note Set.

5.2 AVL Trees

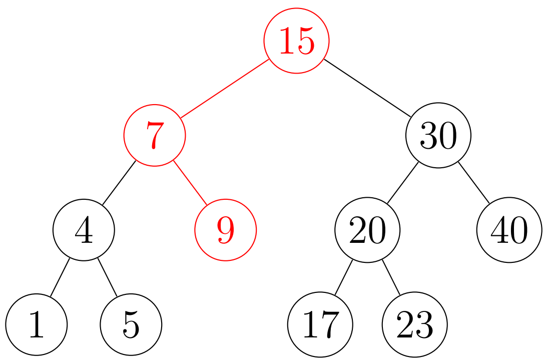
See 310 Note Set.



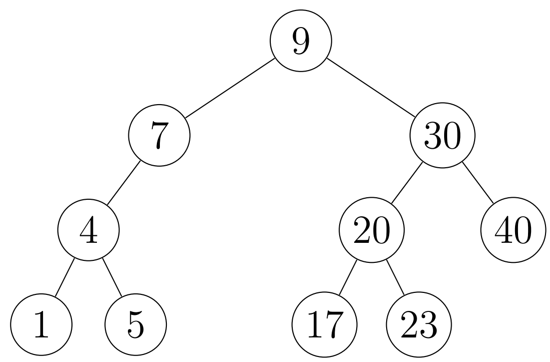
(a) A Binary Search Tree



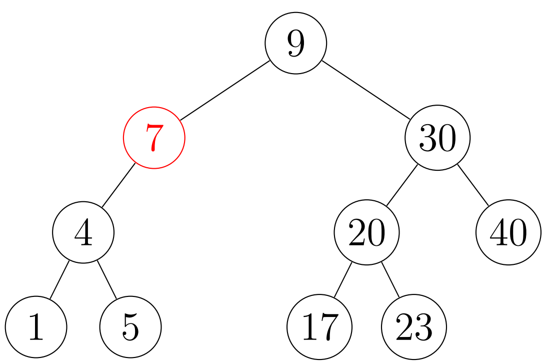
(b) Insertion of a new node (16) into a Binary Search Tree



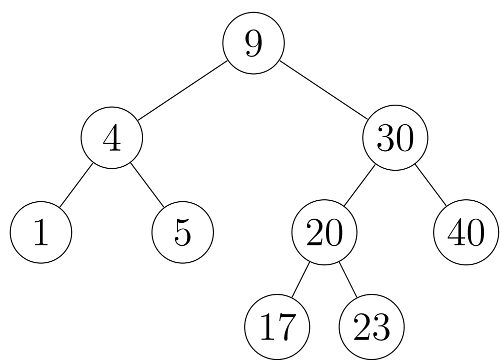
(c) Deletion of a node with two children (15). First step: find the maximum node in the left-sub-tree (lesser elements).



(d) Node 15 is replaced with the extremal node, preserving the BST property



(e) Deletion a node with only one child (7).



(f) Removal is achieved by simply promoting the single child/subtree.

Figure 3: Binary Search Tree Operations. Figure 3(b) depicts the insertion of (16) into the tree in Figure 3(a). Figures 3(c) and 3(d) depict the deletion of a node (15) with two children. Figures 3(e) and 3(f) depict the deletion of a node with only one child (7).

5.3 Red-Black Trees

See 310 Note Set.

6 Optimal Binary Search Trees

See 310 Note Set.

7 Heaps

Definition 1. *A* heap *is a binary tree that satisfies the following properties.*

1. *It is a* full *or* complete *binary tree: all nodes are present except possibly the last row*
2. *If the last row is not full, all nodes are full-to-the-left*
3. *It satisfies the* Heap Property*: every node has a key that is greater than both of its children (max-heap)*

- As a consequence of the Heap Property, the maximal element is always at the root
- Alternatively, we can define a *min-heap*
- Variations: 2-3 heaps, fibonacci heaps, etc.
- A min-heap example can be found in Figure 4

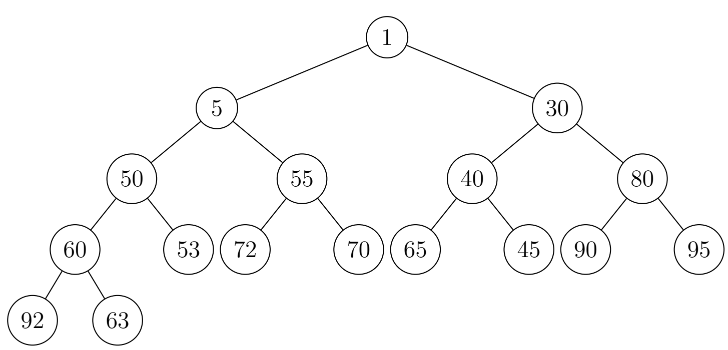


Figure 4: A min-heap

Applications

- Heaps are an optimal implementation of a priority queue
- Used extensively in other algorithms (Heap Sort, Prim's, Dijkstra's, Huffman Coding, etc.) to ensure efficient operation

7.1 Operations

Insert

- Want to preserve the full-ness property and the Heap Property
- Preserve full-ness: add new element at the end of the heap (last row, first free spot on the left)
- Insertion at the end may violate the Heap Property
- Heapify/fix: bubble up inserted element until Heap Property is satisfied
- Analysis: insert is $O(1)$; heapify: $O(d)$

Remove from top

- Want to preserve the full-ness property and the Heap Property
- Preserve full-ness: swap root element with the last element in the heap (lowest row, right-most element)
- Heap property may be violated
- Heapify/fix: bubble new root element down until Heap Property is satisfied
- Analysis: Swap is $O(1)$; heapify: $O(d)$

Others

- Arbitrary remove
- Find
- Possible, but not ideal: Heaps are restricted-access data structures

Analysis

- All operations are $O(d)$
- Since Heaps are *full*, $d = O(\log n)$
- Thus, all operations are $O(\log n)$

7.2 Implementations

Array-Based

- Root is located at index 1
- If a node u is at index i, u's left-child is at $2i$, its right-child is at $2i+1$
- If node u is at index j, its parent is at index $\lfloor \frac{j}{2} \rfloor$
- Alternatively: 0-index array left/right children/parent are at $2n+1, 2n+2$, and $\lfloor \frac{j-1}{2} \rfloor$
- Advantage: easy implementation, all items are contiguous in the array (in fact, a BFS ordering!)
- Disadvantage: Insert operation may force a reallocation, but this can be done in amortized-constant time (though may still have wasted space)

Tree-Based

- Reference-based tree (nodes which contain references to children/parent)
- Parent reference is now *required* for efficiency
- For efficiency, we need to keep track of the *last* element in the tree
- For deletes/inserts: we need a way to find the last element and first "open spot"
- We'll focus on finding the first available open spot as the same technique can be used to find the last element with minor modifications

7.2.1 Finding the first available open spot in a Tree-based Heap

Technique A: numerical technique

- WLOG: assume we keep track of the number of nodes in the heap, n and thus the depth $d = \lfloor \log n \rfloor$
- If $n = 2^{d+1} - 1$ then the tree is full, the last element is all the way to the right, the first available spot is all the way to the left
- Otherwise $n < 2^{d+1} - 1$ and the heap is not full (the first available spot is located at level d, root is at level 0)
- Starting at the root, we want to know if the last element is in the left-subtree or the right subtree

- Let $m = n - (2^d - 1)$, the number of nodes present in level d
- If $m \geq \frac{2^d}{2}$ then the left-sub tree is full at the last level and so the next open spot would be in the right-sub tree
- Otherwise if $m < \frac{2^d}{2}$ then the left-sub tree is not full at the last level and so the next open spot is in the left-sub tree
- Traverse down to the left or right respectively and repeat: the resulting sub-tree will have depth $d - 1$ with $m = m$ (if traversing left) or $m = m - \frac{2^d}{2}$ (if traversing right)
- Repeat until we've found the first available spot
- Analysis: in any case, its $O(d) = O(\log n)$ to traverse from the root to the first open spot

```
INPUT   : A tree-based heap H with n nodes
OUTPUT: The node whose child is the next available open spot in the heap
curr ← T.head
d ← ⌊log n⌋
m ← n
WHILE curr has both children DO
    IF m = 2^{d+1} − 1 THEN
        //remaining tree is full, traverse all the way left
        WHILE curr has both children DO
            curr ← curr.leftChild
        END
    ELSE
        //remaining tree is not full, determine if the next open spot is
          in the left or right sub-tree
        IF m ≥ 2^d/2 THEN
            //left sub-tree is full
            d ← (d − 1)
            m ← (m − 2^d/2)
            curr ← curr.rightChild
        ELSE
            //left sub-tree is not full
            d ← (d − 1)
            m ← m
            curr ← curr.leftChild
        END
    END
END
output curr
```

Algorithm 8: Find Next Open Spot - Numerical Technique

Technique B: Walk Technique

- Alternatively, we can adapt the idea behind the tree walk algorithm to find the next available open spot
- We'll assume that we've kept track of the last node
- If the tree is full, we simply traverse all the way to the left and insert, $O(d)$

- If the last node is a left-child then its parent's right child is the next available spot, finding it is $O(1)$
- Otherwise, we'll need to traverse around the perimeter of the tree until we reach the next open slot

```
INPUT  : A tree-based heap H with n nodes
OUTPUT: The node whose (missing) child is the next available open spot in the heap
d ← ⌊log n⌋
IF n = 2^(d+1) − 1 THEN
    //The tree is full, traverse all the way to the left
    curr ← root
    WHILE curr.leftChild ≠ null DO
        curr ← curr.leftChild
    END
ELSE IF last is a left-child THEN
    //parent's right child is open
    curr ← last.parent
ELSE
    //The open spot lies in a subtree to the right of the last node
    //Walk the tree until we reach it
    curr ← last.parent
    WHILE curr is a right-child DO
        curr ← curr.parent
    END
    //"turn" right
    curr ← curr.parent
    curr ← curr.rightChild
    //traverse all the way left
    WHILE curr.leftChild ≠ null DO
        curr ← curr.leftChild
    END
END
//current node's missing child is the open spot
output curr
```

Algorithm 9: Find Next Open Spot - Walk Technique

7.3 Heap Sort

- If min/max element is always at the top; simply insert all elements, then remove them all!
- Perfect illustration of "Smart data structures and dumb code are a lot better than the other way around"

INPUT : A collection of elements $A = \{a_1, \ldots, a_n\}$
OUTPUT: A collection, A' of elements in A, sorted

$H \leftarrow$ empty heap
$A' \leftarrow$ empty collection
FOREACH $x \in A$ DO
    insert x into H
END
WHILE H *is not empty* DO
    $y \leftarrow$ remove top from H
    Add y to the end of A'
END
output A'

Algorithm 10: Heap Sort

Analysis

- Amortized analysis: insert/remove operations are not constant throughout the algorithm
- On first iteration: insert is $d = O(1)$; on the i-th iteration, $d = O(\log i)$; only on the last iteration is insertion $O(\log n)$
- In total, the insert phase is:

$$\sum_{i=1}^{n} \log i = O(n \log n)$$

- A similar lower bound can be shown
- Same analysis applies to the remove phase:

$$\sum_{i=n}^{1} \log i$$

- In total, $O(n \log n)$

8 Java Collections Framework

Java has support for several data structures supported by underlying tree structures.

- `java.util.PriorityQueue<E>` is a binary-heap based priority queue
  - Priority (keys) based on either *natural ordering* or a provided `Comparator`
  - Guaranteed $O(\log n)$ time for insert (`offer`) and get top (`poll`)
  - Supports $O(n)$ arbitrary `remove(Object)` and search (`contains`) methods
- `java.util.TreeSet<E>`
  - Implements the `SortedSet` interface; makes use of a `Comparator`
  - Backed by `TreeMap`, a red-black tree balanced binary tree implementation
  - Guaranteed $O(\log n)$ time for add, remove, contains operations
  - Default iterator is an in-order traversal

9 Applications

9.1 Huffman Coding

Overview

- Coding Theory is the study and theory of *codes*—schemes for transmitting data
- Coding theory involves efficiently padding out data with redundant information to increase reliability (detect or even correct errors) over a noisy channel
- Coding theory also involves *compressing* data to save space
  - MP3s (uses a form of Huffman coding, but is information lossy)
  - jpegs, mpegs, even DVDs
  - `pack` (straight Huffman coding)
  - `zip`, `gzip` (uses a Ziv-Lempel and Huffman compression algorithm)

Basics

- Let Σ be a fixed *alphabet* of size n
- A *coding* is a mapping of this alphabet to a collection of binary *codewords*,

$$\Sigma \to \{0,1\}^*$$

- A *block encoding* is a *fixed length encoding* scheme where all codewords have the same length (example: ASCII); requires $\lceil \log_2 n \rceil$ length codes
- Not all symbols have the same frequency, alternative: *variable length encoding*
- Intuitively: assign shorter codewords to more frequent symbols, longer to less frequent symbols
- Reduction in the overall *average* codeword length
- Variable length encodings must be *unambiguous*
- Solution: *prefix free codes*: a code in which no *whole* codeword is the prefix of another (other than itself of course).
- Examples:
  - $\{0, 01, 101, 010\}$ is not a prefix free code.
  - $\{10, 010, 110, 0110\}$ is a prefix free code.
- A simple way of building a prefix free code is to associate codewords with the *leaves* of a binary tree (not necessarily full).
- Each edge corresponds to a bit, 0 if it is to the left sub-child and 1 to the right sub-child.
- Since no simple path from the root to any leaf can continue to another leaf, then we are guaranteed a prefix free coding.
- Using this idea along with a greedy encoding forms the basis of *Huffman Coding*

Steps

- Consider a precomputed relative frequency function:

$$\text{freq} : \Sigma \to [0, 1]$$

- Build a collection of *weighted* trees T_x for each symbol $x \in Sigma$ with $wt(T_x) = \text{freq}(x)$
- Combine the two least weighted trees via a new node; associate a new weight (the sum of the weights of the two subtrees)
- Keep combining until only one tree remains
- The tree constructed in Huffman's algorithm is known as a *Huffman Tree* and it defines a *Huffman Code*

```
INPUT   : An alphabet of symbols, Σ with relative frequencies, freq(x)
OUTPUT: A Huffman Tree
H ← new min-heap
FOREACH x ∈ Σ DO
    T_x ← single node tree
    wt(T_x) ← freq(x)
    insert T_x into H
END
WHILE size of H > 1 DO
    T_r ← new tree root node
    T_a ← H.getMin
    T_b ← H.getMin
    T_r.leftChild ← T_a
    T_r.rightChild ← T_b
    wt(r) ← wt(T_a) + wt(T_b)
    insert T_r into H
END
output H.getMin
```

Algorithm 11: Huffman Coding

9.1.1 Example

Construct the Huffman Tree and Huffman Code for a file with the following content.

character	A	B	C	D	E	F	G
frequency	0.10	0.15	0.34	.05	.12	.21	.03

- Average codeword length:

$$.10 \cdot 3 + .15 \cdot 3 + .34 \cdot 2 + .05 \cdot 4 + .12 \cdot 3 + .21 \cdot 2 + .03 \cdot 4 = 2.53$$

- Compression ratio:

$$\frac{(3 - 2.53)}{3} = 15.67\%$$

- In general, for text files, `pack` (Huffman Coding), claims an average compression ratio of 25-40%.
- Degenerative cases:

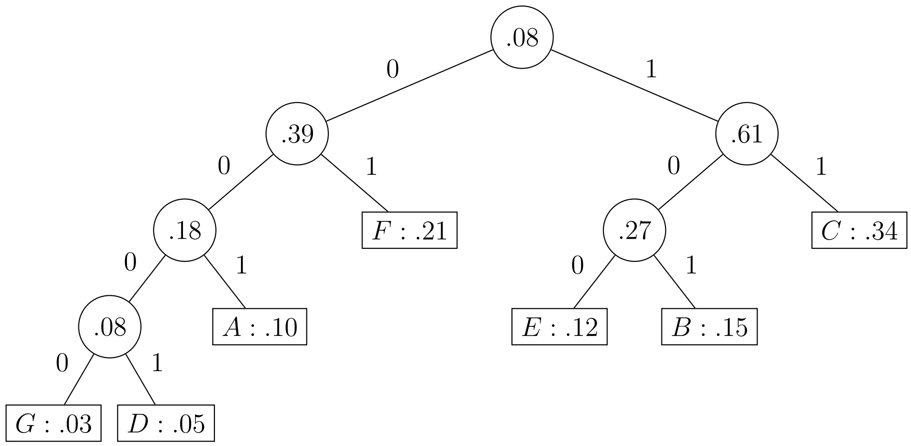


Figure 5: Huffman Tree

- When the probability distribution is uniform: $p(x) = p(y)$ for all $x, y \in \Sigma$
- When the probability distribution follows a fibonacci sequence (the sum of each of the two smallest probabilities is less than equal to the next highest probability for all probabilities)

A Stack-based Traversal Simulations

A.1 Preorder

Simulation of a stack-based preorder traversal of the binary tree in Figure 1.

`push` a

(enter loop)
`pop`, $node = a$
`push` c
`push` b
print a

(enter loop)
`pop`, $node = b$
`push` e
`push` d
print b

(enter loop)
`pop`, $node = d$
`push` h
`push` g
print d

(enter loop)
`pop`, $node = g$
`push` m
`push` l
print g

(enter loop)
`pop`, $node = l$
(no push)
(no push)
print l

(enter loop)
`pop`, $node = m$
(no push)
`push` r
print m

(enter loop)
`pop`, $node = r$
(no push)
(no push)
print r

(enter loop)
`pop`, $node = h$
`push` n
(no push)
print h

(enter loop)
`pop`, $node = n$
(no push)
(no push)
print n

(enter loop)
`pop`, $node = e$
`push` i
(no push)
print e

(enter loop)
`pop`, $node = i$
(no push)
`push` o
print i

(enter loop)
`pop`, $node = o$
(no push)
(no push)
print o

(enter loop)
`pop`, $node = c$
`push` f
(no push)
print c

(enter loop)
`pop`, $node = f$
`push` k
`push` j
print f

(enter loop)
`pop`, $node = j$
`push` q
`push` p
print j

(enter loop)
`pop`, $node = p$
(no push)

(no push)
print p

(enter loop)
pop, $node = q$

(no push)
(no push)
print q

(enter loop)
pop, $node = k$

(no push)
(no push)
print k

A.2 Inorder

Simulation of a stack-based inorder traversal of the binary tree in Figure 1.

(enter loop, $u = a$)
push a
update $u = b$

(enter loop, $u = b$)
push b
update $u = d$

(enter loop, $u = d$)
push d
update $u = g$

(enter loop, $u = g$)
push g
update $u = l$

(enter loop, $u = l$)
push l
update $u =$ null

(enter loop,
$u =$ null)
pop l, update $u = l$
process l
update $u =$ null

(enter loop,
$u =$ null)
pop g, update
$u = g$
process g
update $u = m$

(enter loop, $u = m$)
push m
update $u = r$

(enter loop, $u = r$)
push r
update $u =$ null

(enter loop,
$u =$ null)
pop r, update
$u = r$
process r
update $u =$ null

(enter loop,
$u =$ null)
pop m, update
$u = m$
process m
update $u =$ null

(enter loop,
$u =$ null)
pop d, update
$u = d$
process d
update $u = h$

(enter loop, $u = h$)
push h
update $u =$ null

(enter loop,
$u =$ null)
pop h, update
$u = h$
process h
update $u = n$

(enter loop, $u = n$)
push n
update $u =$ null

(enter loop,
$u =$ null)
pop n, update
$u = n$
process n
update $u =$ null

(enter loop,
$u =$ null)
pop b, update
$u = b$
process b
update $u = e$

(enter loop, $u = e$)
push e
update $u =$ null

(enter loop,
$u =$ null)
pop e, update
$u = e$
process e
update $u = i$

(enter loop, $u = i$)
push i
update $u = o$

(enter loop, $u = o$)
push o
update $u =$ null

(enter loop,
$u =$ null)
pop o, update
$u = o$
process o
update $u =$ null

(enter loop,
$u =$ null)
pop i, update
$u = i$
process i
update $u =$ null

(enter loop,
$u =$ null)
pop i, update
$u = i$
process i
update $u =$ null

(enter loop,
$u =$ null)
pop a, update
$u = a$
process a
update $u = c$

(enter loop, $u = c$)
push c
update $u =$ null

(enter loop,
$u =$ null)
pop c, update
$u = c$
process c
update $u = f$

(enter loop, $u = f$)
push f
update $u = j$

(enter loop, $u = j$)
push j
update $u = p$

(enter loop, $u = p$)
push p
update $u =$ null

(enter loop,
$u =$ null)
pop p, update
$u = p$
process p
update $u =$ null

(enter loop,
$u =$ null)
pop j, update
$u = j$
process j
update $u = q$

(enter loop, $u = q$)
push q
update $u =$ null

(enter loop,
$u =$ null)
pop q, update
$u = q$
process q
update $u =$ null

(enter loop,
$u =$ null)
pop f, update
$u = f$
process f
update $u = k$

(enter loop, $u = k$)
push k
update $u =$ null

(enter loop,
$u =$ null)
pop k, update
$u = k$
process k
update $u =$ null

(done)

A.3 Postorder

Simulation of a stack-based postorder traversal of the binary tree in Figure 1:

$prev = \texttt{null}$
$\texttt{push}\ \ a$

(enter loop)
update $curr = (a)$
check: $prev = \texttt{null}$
$\texttt{push}\ \ (b)$
update $prev = a$

(enter loop)
update $curr = (b)$
check:
$prev.leftChild = curr$
$((b).leftChild = (b))$
$\texttt{push}\ \ (d)$
update $prev = b$

(enter loop)
update $curr = (d)$
check:
$prev.leftChild = curr$
$((d).leftChild = (d))$
$\texttt{push}\ \ (g)$
update $prev = d$

(enter loop)
update $curr = (g)$
check:
$prev.leftChild = curr$
$((g).leftChild = (g))$
$\texttt{push}\ \ (l)$
update $prev = g$

(enter loop)
update $curr = (l)$
check:
$prev.leftChild = curr$
$((l).leftChild = (l))$
(noop)
update $prev = l$

(enter loop)
update $curr = (l)$
check:
$prev.rightChild = curr$
$(null.rightChild = (l))$
process l
update $prev = l$

(enter loop)
update $curr = (g)$
check:
$prev.rightChild = curr$
$(null.rightChild = (g))$
check:
$curr.leftChild = prev$
$((l).leftChild = (l))$
$\texttt{push}\ \ (m)$
update $prev = g$

(enter loop)
update $curr = (m)$
check:
$prev.rightChild = curr$
$((m).rightChild = (m))$
$\texttt{push}\ \ (r)$
update $prev = m$

(enter loop)
update $curr = (r)$
check:
$prev.leftChild = curr$
$((r).leftChild = (r))$
(noop)
update $prev = r$

(enter loop)
update $curr = (r)$
check:
$prev.rightChild = curr$
$(null.rightChild = (r))$
process r
update $prev = r$

(enter loop)
update $curr = (m)$
check:
$prev.rightChild = curr$
$(null.rightChild = (m))$
check:
$curr.leftChild = prev$
$((r).leftChild = (r))$
update $prev = m$

(enter loop)
update $curr = (m)$
check:
$prev.rightChild = curr$
$(null.rightChild = (m))$
process m
update $prev = m$

(enter loop)
update $curr = (g)$
check:
$prev.rightChild = curr$
$(null.rightChild = (g))$
process g
update $prev = g$

(enter loop)
update $curr = (d)$
check:
$prev.rightChild = curr$
$((m).rightChild = (d))$
check:
$curr.leftChild = prev$
$((g).leftChild = (g))$
$\texttt{push}\ \ (h)$
update $prev = d$

(enter loop)
update $curr = (h)$
check:
$prev.rightChild = curr$
$((h).rightChild = (h))$
$\texttt{push}\ \ (n)$
update $prev = h$

(enter loop)
update $curr = (n)$
check:
$prev.rightChild = curr$
$((n).rightChild = (n))$
(noop)
update $prev = n$

(enter loop)
update $curr = (n)$
check:
$prev.rightChild = curr$
$(null.rightChild = (n))$
process n
update $prev = n$

(enter loop)
update $curr = (h)$
check:
$prev.rightChild = curr$
$(null.rightChild = (h))$
process h
update $prev = h$

(enter loop)
update $curr = (d)$
check:
$prev.rightChild = curr$
$((n).rightChild = (d))$
process d
update $prev = d$

(enter loop)
update $curr = (b)$
check:
$prev.rightChild = curr$
$((h).rightChild = (b))$
check:
$curr.leftChild = prev$
$((d).leftChild = (d))$
$\texttt{push}\ \ (e)$
update $prev = b$

(enter loop)
update $curr = (e)$
check:
$prev.rightChild = curr$
$((e).rightChild = (e))$
$\texttt{push}\ \ (i)$
update $prev = e$

(enter loop)
update $curr = (i)$
check:
$prev.rightChild = curr$

$((i).rightChild = (i))$
`push` (o)
update $prev = i$

(enter loop)
update $curr = (o)$
check:
$prev.leftChild = curr$
$((o).leftChild = (o))$
(noop)
update $prev = o$

(enter loop)
update $curr = (o)$
check:
$prev.rightChild = curr$
$(null.rightChild = (o))$
process o
update $prev = o$

(enter loop)
update $curr = (i)$
check:
$prev.rightChild = curr$
$(null.rightChild = (i))$
check:
$curr.leftChild = prev$
$((o).leftChild = (o))$
update $prev = i$

(enter loop)
update $curr = (i)$
check:
$prev.rightChild = curr$
$(null.rightChild = (i))$
process i
update $prev = i$

(enter loop)
update $curr = (e)$
check:
$prev.rightChild = curr$
$(null.rightChild = (e))$
process e
update $prev = e$

(enter loop)
update $curr = (b)$
check:
$prev.rightChild = curr$
$((i).rightChild = (b))$
process b
update $prev = b$

(enter loop)
update $curr = (a)$
check:
$prev.rightChild = curr$
$((e).rightChild = (a))$
check:
$curr.leftChild = prev$
$((b).leftChild = (b))$
`push` (c)
update $prev = a$

(enter loop)
update $curr = (c)$
check:
$prev.rightChild = curr$
$((c).rightChild = (c))$
`push` (f)
update $prev = c$

(enter loop)
update $curr = (f)$
check:
$prev.rightChild = curr$
$((f).rightChild = (f))$
`push` (j)
update $prev = f$

(enter loop)
update $curr = (j)$
check:
$prev.leftChild = curr$
$((j).leftChild = (j))$
`push` (p)
update $prev = j$

(enter loop)
update $curr = (p)$
check:
$prev.leftChild = curr$
$((p).leftChild = (p))$
(noop)
update $prev = p$

(enter loop)
update $curr = (p)$
check:
$prev.rightChild = curr$
$(null.rightChild = (p))$
process p
update $prev = p$

(enter loop)
update $curr = (j)$
check:
$prev.rightChild = curr$
$(null.rightChild = (j))$
check:
$curr.leftChild = prev$
$((p).leftChild = (p))$
`push` (q)
update $prev = j$

(enter loop)
update $curr = (q)$
check:
$prev.rightChild = curr$
$((q).rightChild = (q))$
(noop)
update $prev = q$

(enter loop)
update $curr = (q)$
check:
$prev.rightChild = curr$
$(null.rightChild = (q))$
process q
update $prev = q$

(enter loop)
update $curr = (j)$
check:
$prev.rightChild = curr$
$(null.rightChild = (j))$
process j
update $prev = j$

(enter loop)
update $curr = (f)$
check:
$prev.rightChild = curr$
$((q).rightChild = (f))$
check:
$curr.leftChild = prev$
$((j).leftChild = (j))$
`push` (k)
update $prev = f$

(enter loop)
update $curr = (k)$
check:
$prev.rightChild = curr$
$((k).rightChild = (k))$
(noop)
update $prev = k$

(enter loop)
update $curr = (k)$
check:
$prev.rightChild = curr$
$(null.rightChild = (k))$
process k
update $prev = k$

(enter loop)
update $curr = (f)$
check:
$prev.rightChild = curr$
$(null.rightChild = (f))$
process f
update $prev = f$

(enter loop)
update $curr = (c)$
check:
$prev.rightChild = curr$
$((k).rightChild = (c))$
process c
update $prev = c$

(enter loop)
update $curr = (a)$
check:
$prev.rightChild = curr$
$((f).rightChild = (a))$
process a
update $prev = a$