1 Axioms of Zermelo-Frankel Set Theory

1.1 Axiom of Extensionality

- Two sets are equal if they have the same elements

$$\forall X \forall Y \left[\forall z \left(z \in X \iff z \in Y \right) \Rightarrow X = Y \right]$$

1.2 Axiom of Regularity

- Every non-empty set X contains a member Y such that X, Y are disjoint sets.

$$\forall X \left[\exists a (a \in x) \Rightarrow \exists y (y \in X \land \neg \exists z (z \in y \land z \in x)) \right]$$

1.3 Axiom of Schema of Specification

- Any definable subclass of a set is a set
- AKA: Axiom of Schema Separation or of Restricted Comprehension
- Avoids Russell's Paradox
- Let ϕ be a poperty which may characterize the elements x of Z, then there is a subset $Y \subseteq Z$ containing those $x \in Z$ which satisfy ϕ

$$\forall z \forall w_1, \ldots, w_n \exists y \forall x \left[x \in y \iff (x \in z \land \phi(x, z, w_1, \ldots, w_n) \right]$$

1.4 Axiom of Pairing

- If X, Y are sets, then there is a set which contains X, Y as elements

$$\forall X \forall Y \exists Z \left[x \in z \land y \in z \right]$$

1.5 Axiom of Union

- For any set F there is a set A containing every set that is a member of some member of F

$$\forall F \exists A \forall Y \forall x \left[(x \in Y \land Y \in F) \Rightarrow x \in A \right]$$

1.6 Axiom Schema of Replacement

- If $f : A \to B$ is a definable function with the domain A being a set and $f(x)$ is a set for all x in the domain, then teh range of f is a subclass of a set (with a paradox-avoiding restriction)

- $\exists! x$ is the *uniqueness* quantifier. Defined by Kleene (1952), it is true if there exists *exactly one* element in the universe of discourse that satisfies the predicate; formally:

$$\exists! x P(x) \equiv \exists x \, [P(x) \wedge \forall y (P(y) \to x = y)]$$

- Note: there is also a $\exists!!$ quantifier that is used for distinctness

- Let $\phi(x, y, A, w_1, \ldots, w_n)$ be a formula, then

$$\forall A \forall w_1, \ldots, w_n \, [\forall x (x \in A \Rightarrow \exists! y \phi() \Rightarrow \exists B \forall x (x \in A \Rightarrow \exists y (y \in B \wedge \phi))]$$

1.7 Axiom of Infinity

- There exists a set X such that the empty set $\emptyset$ is a member of X and, whenever a set Y is a member of X, then $Y \cup \{Y\}$ is also a member of X

$$\exists X \, [\emptyset \in X \wedge \forall y (y \in X \Rightarrow (y \cup \{y\}) \in X)]$$

1.8 Axiom of the Power Set

- For any set X there is a set Y which is a superset of the power set of X, $\mathcal{P}(X)$

$$\forall X \exists Y \exists Z \, [Z \subseteq x \Rightarrow Z \in Y]$$

1.9 Well-Ordering Theorem

- For any set X, there is a binary relation R which well-orders X

- Equivalent to the Axiom of Choice: Let X be a set whose members are non-empty, then there exists a function $f : X \to B$ where B is the union of some members of X (a choice function) such that for all $Y \in X, f(B) \in B$

- Axioms 1 – 8 imply 9 for finite sets
- For infinite sets, AC is non-constructive and leads to some statements being *independent*