Sets

Computer Science & Engineering 235

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Introduction I

We've already implicitly dealt with sets (integers, $\mathbb{Z}$; rationals ($\mathbb{Q}$) etc.) but here we will develop more fully the definitions, properties and operations of sets.

Definition

A *set* is an unordered collection of unique, but similar objects.

Sets are fundamental discrete structures that form the basis of more complex discrete structures like graphs.

Contrast this definition with the one in the book (compare *multi-set, ordered set*, etc).

Definition

Introduction II

The objects in a set are called *elements* or *members* of a set. A set is said to *contain* its elements.

Recall the notation: for a set A, an element x we write

$$x \in A$$

if A contains x and

$$x \notin A$$

otherwise.

Terminology I

Definition

Two sets, A and B are *equal* if they contain the same elements. In this case we write $A = B$.

Example

Terminology II

$$\{2, 3, 5, 7\} = \{3, 2, 7, 5\}$$

since a set is *unordered*. Also,

$$\{2, 3, 5, 7\} = \{2, 2, 3, 3, 5, 7\}$$

since a set contains *unique* elements (i.e. it is not a *multi-set*. However,

$$\{2, 3, 5, 7\} \neq \{2, 3\}$$

We've already seen *set builder* notation:

$$O = \{x \mid (x \in \mathbb{Z}) \wedge (x = 2k \text{ for some } k \in \mathbb{Z})\}$$

should be read O is the set that contains all x such that x is an integer and x is even.

Terminology III

A set can also be represented graphically using a *Venn diagram*.

Venn Diagram
Example

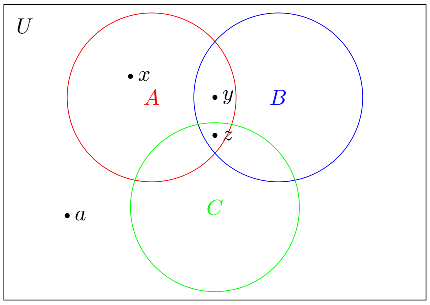


Figure : Venn Diagram

More Terminology & Notation I

A set that has no elements is referred to as the *empty set* or *null set* and is denoted $\emptyset$.

A *singleton* set is a set that has only one element. We usually write $\{a\}$. Note the different: brackets indicate that the object is a *set* while a without brackets is an *element*.

The subtle difference also exists with the empty set: that is

$$\emptyset \neq \{\emptyset\}$$

The first is a set, the second is a set containing a set.

More Terminology & Notation II

Definition

A is said to be a subset of B and we write

$$A \subseteq B$$

if and only if every element of A is also an element of B.

That is, we have an equivalence:

$$A \subseteq B \iff \forall x (x \in A \rightarrow x \in B)$$

More Terminology & Notation III

Theorem

For any set S,

- $\emptyset \subseteq S$ *and*
- $S \subseteq S$

The proof is in the book—note that it is an excellent example of a vacuous proof!

More Terminology & Notation IV

Definition

A set A that is a subset of B is called a *proper subset* if $A \neq B$. That is, there is some element $x \in B$ such that $x \notin A$. In this case we write $A \subset B$ or to be even more definite we write

$$A \subsetneq B$$

Example

Let $A = \{2\}$. Let $B = \{x \mid (x \leq 100) \wedge (x \text{ is prime})\}$. Then $A \subsetneq B$.

More Terminology & Notation V

Sets can be elements of other sets.

Example

$$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

and

$$\{\{1\}, \{2\}, \{3\}\}$$

are sets with sets for elements.

Example 1

Let

$$A = \{x | x \in \mathbb{Z}, x \text{ is even }\}$$

Is $A \subseteq \mathbb{Z}$?
Yes, every even integer is also an integer
Is $\mathbb{Z} \subseteq A$?
No, since for $x = 3$, $x \in \mathbb{Z}$, but $x \notin A$
Is $10 \in A$?
Yes, 10 is even, it is an element of A
Is $\{10\} \in A$?
No, $\{10\}$ is a *set*, $\{10\} \subseteq A$, but it is *not* an *element* of A

Example 2

Let

$$A = \{2, 3, 5\}$$

$$B = \{2, 3, 5, 10\}$$

Is $A \subseteq B$?
Yes, every element in A is in B
Is $A \subsetneq B$?
Yes, since $A \subseteq B$ and we have an element 10 that is not in A, i.e. $A \neq B$

Example 3

Let

$$B = \{2, 3, 5, 10\}$$

$$C = \{2, 3, 5, \{10\}\}$$

Which is true: $B = C$, $B \subseteq C$, $C \subseteq B$?
None; B is a set containing only elements; C is a set containing elements and a set
Is $\{10\} \in C$?
Yes, the set containing the element 10 is an element of the set C.
Is $\{\{10\}\} \subseteq C$?
Yes, as previously established, the element $\{10\} \in C$ which is the only element in $\{\{10\}\}$

Cardinality

Definition

Definition

If there are exactly n distinct elements in a set S, with n a nonnegative integer, we say that S is a finite set and the cardinality of S is n. Notationally, we write

$$|S| = n$$

Definition

A set that is not finite is said to be *infinite*.

Cardinality

Examples

Example

Recall the set $B = \{x \mid (x \leq 100) \wedge (x \text{ is prime})\}$, its cardinality is

$$|B| = 25$$

since there are 25 primes less than 100.

- The cardinality of the empty set is zero:

$$|\emptyset| = 0$$

- The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all infinite.

Cardinality for Infinite Sets I

- The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all called *countable*
- Iintuitively: we can order their elements so that all of them are enumerated

$$\begin{aligned} \mathbb{N} &= \{0, 1, 2, 3, \dots\} \\ \mathbb{Z} &= \{0, 1, -1, 2, -2, 3, -3, \dots\} \\ \mathbb{Q} &= \{0, \tfrac{1}{1}, \tfrac{1}{2}, \tfrac{2}{1}, \tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}, \dots\} \end{aligned}$$

- The set $\mathbb{R}$ is *uncountably* infinite since no such enumeration is possible

Cardinality for Infinite Sets II

- There exist different magnitudes of *infiniteness*
- Notions first explored by Georg Cantor (1874)
- Intuitively: there are "more" reals than there are integers
- (Counter) intuitively: there are just as many integers as there are rational numbers!
- When dealing with infinite sets, cardinality is measured in terms of the existence (or non-existence) of *bijective functions* between their elements

Cardinality for Infinite Sets III

- Notations: cardinaltiy of $\mathbb{Z}$ is $\aleph_0$, cardinality of $\mathbb{R}$ is $\aleph_1$
- $\aleph$ is read: "aleph" (from the hebrew alphabet)
- Continuum Hypothesis: there exists no set of intermediate size between $\mathbb{Z}, \mathbb{R}$
- Proof/disproof is not possible: Shown to be independent (of Zermelo-Fraenkel and Axiom of Choice) by Gödel (1940)
- There is an infinite hierarchy of infinite sets, $\aleph_0, \aleph_1, \aleph_2, \ldots$

Proving Equivalence I

You may be asked to show that a set is a subset, proper subset or equal to another set. To do this, use the equivalence discussed before:

$$A \subseteq B \iff \forall x(x \in A \rightarrow x \in B)$$

To show that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B. Any proof method could be used.

To show that $A \subsetneq B$ you must show that A is a subset of B just as before. But you must also show that

$$\exists x((x \in B) \wedge (x \notin A))$$

Finally, to show two sets equal, it is enough to show (much like an equivalence) that $A \subseteq B$ and $B \subseteq A$ independently.

Proving Equivalence II

Logically speaking this is showing the following quantified statements:

$$\big(\forall x(x \in A \rightarrow x \in B)\big) \wedge \big(\forall x(x \in B \rightarrow x \in A)\big)$$

We'll see an example later.

The Power Set I

Definition

The *power set* of a set S, denoted $\mathcal{P}(S)$ is the set of all subsets of S.

Example

Let $A = \{a, b, c\}$ then the power set is

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Note that the empty set and the set itself are always elements of the power set. This follows from Theorem 1 (Rosen, p81).

The Power Set II

The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set.

Fact

Let S be a set such that $|S| = n$, then

$$|\mathcal{P}(S)| = 2^n$$

Cartesian Products I

Sometimes we may need to consider ordered collections.

Definition

The *ordered n-tuple* $(a_1, a_2, \dots, a_n)$ is the ordered collection with the a_i being the i-th element for $i = 1, 2, \dots, n$.

Two ordered n-tuples are equal if and only if for each $i = 1, 2, \dots, n$, $a_i = b_i$.

For $n = 2$, we have *ordered pairs*.

Definition

Cartesian Products II

Let A and B be sets. The *Cartesian product* of A and B denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$$

A subset of a Cartesian product, $R \subseteq A \times B$ is called a *relation*. We will talk more about relations in the next set of slides.

Note that $A \times B \neq B \times A$ unless $A = \emptyset$ or $B = \emptyset$ or $A = B$. Can you find a counter example to prove this?

Cartesian Products III

Cartesian products can be generalized for any n-tuple.

Definition

The *Cartesian product* of n sets, $A_1, A_2, \dots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

Notation With Quantifiers

Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language.

Now we can simplify things using set notation!

Example

$$\forall x \in \mathbb{R}(x^2 \geq 0)$$
$$\exists x \in \mathbb{Z}(x^2 = 1)$$

Or you can mix quantifiers:

$$\forall a, b, c \in \mathbb{R} \exists x \in \mathbb{C}(ax^2 + bx + c = 0)$$

Set Operations

Just as arithmetic operators can be used on pairs of numbers, there are operators that can act on sets to give us new sets.

Set Operators

Union

Definition

The *union* of two sets A and B is the set that contains all elements in A, B or both. We write

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

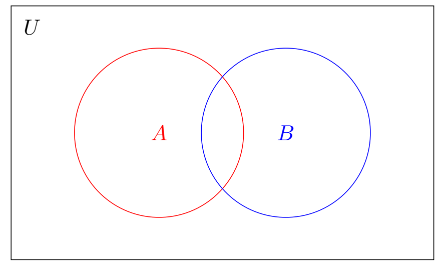
Set Operators
Intersection

Definition

The *intersection* of two sets A and B is the set that contains all elements that are elements of *both* A *and* B We write

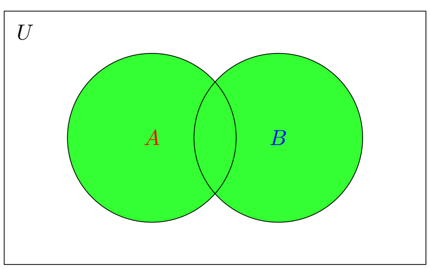
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Set Operators
Venn Diagram Example



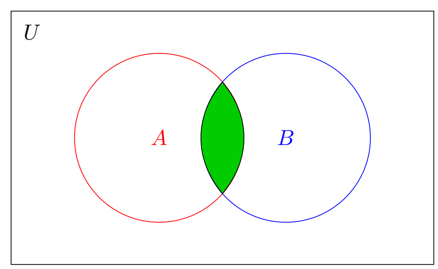
Sets A and B

Set Operators
Venn Diagram Example: Union



Union, $A \cup B$

Set Operators
Venn Diagram Example: Intersection



Intersection, $A \cap B$

Disjoint Sets

Definition

Two sets are said to be *disjoint* if their intersection is the empty set: $A \cap B = \emptyset$

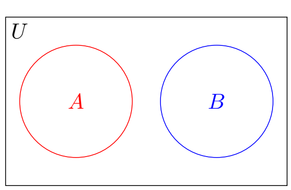


Figure : Two disjoint sets A and B.

Set Difference

Definition

The *difference* of sets A and B, denoted by $A \setminus B$ (or $A - B$) is the set containing those elements that are in A but not in B.

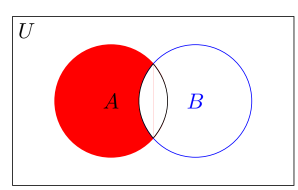


Figure : Set Difference, $A \setminus B$

Set Complement

Definition

The *complement* of a set A, denoted $\bar{A}$, consists of all elements *not* in A. That is, the difference of the universal set and A; $U \setminus A$.

$$\bar{A} = \{x \mid x \notin A\}$$

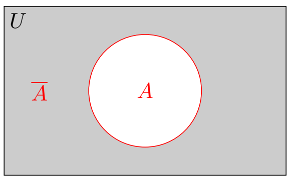


Figure : Set Complement, $\overline{A}$

DeMorgan's Laws for Sets

De Morgan's Laws for sets:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Set Identities

There are analogs of all the usual laws for set operations.

See your course cheat sheet for a list.

Proving Set Equivalences

Recall that to *prove* such an identity, one must show that

1. The left hand side is a subset of the right hand side.
2. The right hand side is a subset of the left hand side.
3. Then conclude that they are, in fact, equal.

The book proves several of the standard set identities. We'll give a couple of different examples here.

Proving Set Equivalences

Example I

Let $A = \{x \mid x \text{ is even}\}$ and $B = \{x \mid x \text{ is a multiple of } 3\}$ and $C = \{x \mid x \text{ is a multiple of } 6\}$. Show that

$$A \cap B = C$$

Proof.

($A \cap B \subseteq C$): Let $x \in A \cap B$. Then x is a multiple of 2 *and* x is a multiple of 3, therefore we can write $x = 2 \cdot 3 \cdot k$ for some integer k. Thus $x = 6k$ and so x is a multiple of 6 and $x \in C$.

($C \subseteq A \cap B$): Let $x \in C$. Then x is a multiple of 6 and so $x = 6k$ for some integer k. Therefore $x = 2(3k) = 3(2k)$ and so $x \in A$ and $x \in B$. It follows then that $x \in A \cap B$ by definition of intersection, thus $C \subseteq A \cap B$.

We conclude that $A \cap B = C$ □

Proving Set Equivalences

Example II

An alternative prove uses *membership tables* where an entry is 1 if it a chosen (but fixed) element is in the set and 0 otherwise.

Example

(Exercise 13, p95): Show that

$$\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$$

Proving Set Equivalences

Example II Continued

A	B	C	$A\cap B\cap C$	$\overline{A\cap B\cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A}\cup\bar{B}\cup\bar{C}$
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	0	1
0	1	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	0	1	0	1
1	1	0	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0

Since the columns are equivalent, we conclude that indeed,

$$\overline{A\cap B\cap C} = \bar{A}\cup\bar{B}\cup\bar{C}$$

Generalized Unions & Intersections I

In the previous example we showed that De Morgan's Law generalized to unions involving 3 sets. Indeed, for any finite number of sets, De Morgan's Laws hold.

Moreover, we can generalize set operations in a straightforward manner to any finite number of sets.

Definition

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Generalized Unions & Intersections II

Definition

The *intersection* of a collection of sets is the set that contains those elements that are members of *every* set in the collection.

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Computer Representation of Sets I

There really aren't ways to represent *infinite* sets by a computer since a computer is has a finite amount of memory (unless of course, there is a finite *representation*).

If we assume that the universal set U is finite, however, then we can easily and efficiently represent sets by *bit vectors*.

Specifically, we force an ordering on the objects, say

$$U = \{a_1, a_2, \ldots, a_n\}$$

For a set $A \subseteq U$, a bit vector can be defined as

$$b_i = \begin{cases} 0 & \text{if } a_i \notin A \\ 1 & \text{if } a_i \in A \end{cases}$$

for $i = 1, 2, \ldots, n$.

Computer Representation of Sets II

Example

Let $U = \{0,1,2,3,4,5,6,7\}$ and let $A = \{0,1,6,7\}$ Then the bit vector representing A is

$$1100\ 0011$$

What's the empty set? What's U?

Set operations become almost trivial when sets are represented by bit vectors.

In particular, the bit-wise OR corresponds to the union operation. The bit-wise AND corresponds to the intersection operation.

Example

Computer Representation of Sets III

Let U and A be as before and let $B = \{0,4,5\}$ Note that the bit vector for B is 1000 1100. The union, $A\cup B$ can be computed by

$$1100\ 0011 \vee 1000\ 1100 = 1100\ 1111$$

The intersection, $A\cap B$ can be computed by

$$1100\ 0011 \wedge 1000\ 1100 = 1000\ 0000$$

What sets do these represent?

Note: If you want to represent *arbitrarily* sized sets, you can still do it with a computer—how?

Set Operation Algorithms I

Power Set: Recursive

Algorithm (RecursivePowerSet)

INPUT : $S = \{s_1, \ldots, s_n\}$, $A \subseteq S$, $index$
OUTPUT : *The power set,* $\mathcal{P}$

IF $index > n$ THEN
    output A
ELSE
    RECURSIVEPOWERSET$(S, A \cup \{s_{index}, index + 1\})$
    RECURSIVEPOWERSET$(S, A, index + 1\})$
END

Set Operation Algorithms I

Power Set: Iterative

Algorithm (IterativePowerSet)

Set Operation Algorithms II

Power Set: Iterative

INPUT : $S = \{s_1, \ldots, s_n\}$
OUTPUT : *The power set,* $\mathcal{P}$

$\mathcal{P} \leftarrow \emptyset$
FOR $i = 0, \ldots, 2^n - 1$ DO
    $\vec{b} = b_1 b_2 \cdots b_n \leftarrow$ *convert* i *to binary*
    $A \leftarrow \emptyset$
    FOR $j = 1, \ldots, n$ DO
        IF b_j THEN
            $A \leftarrow A \cup \{s_j\}$
        END
    END
    $\mathcal{P} \leftarrow \mathcal{P} \cup \{A\}$
END
output $\mathcal{P}$