Relations

Computer Science & Engineering 235: Discrete Mathematics

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Introduction

Recall that a relation between elements of two sets is a subset of their Cartesian product (of ordered pairs).

Definition

A *binary relation* from a set A to a set B is a subset

$$R \subseteq A \times B = \{(a,b) \mid a \in A, b \in B\}$$

Note the difference between a relation and a function: in a relation, each $a \in A$ can map to multiple elements in B. Thus, relations are generalizations of functions.

If an ordered pair $(a,b) \in R$ then we say that a is *related* to b. We may also use the notation aRb and $a \not R b$.

Relations

To represent a relation, you can enumerate every element in R.

Example

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$ let R be a relation from A to B as follows:

$$R = \{(a_1,b_1),(a_1,b_2),(a_1,b_3),(a_2,b_1),(a_3,b_1),(a_3,b_2),(a_3,b_3),(a_5,b_1)\}$$

You can also represent this relation graphically.

Relations

Graphical View

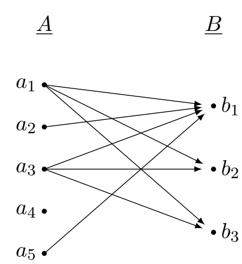


Figure : Graphical Representation of a Relation

Relations

On a Set

Definition

A relation on the set A is a relation from A to A. I.e. a subset of $A \times A$.

Example

The following are binary relations on $\mathbb{N}$:

$$R_1 = \{(a,b) \mid a \le b\}$$

$$R_2 = \{(a,b) \mid a,b \in \mathbb{N}, \frac{a}{b} \in \mathbb{Z}\}$$

$$R_3 = \{(a,b) \mid a,b \in \mathbb{N}, a - b = 2\}$$

Exercise: Give some examples of ordered pairs $(a,b) \in \mathbb{N}^2$ that are not in each of these relations.

Reflexivity

Definition

There are several properties of relations that we will look at. If the ordered pairs (a,a) appear in a relation on a set A for every $a \in A$ then it is called reflexive.

Definition

A relation R on a set A is called *reflexive* if

$$\forall a \in A \big((a,a) \in R\big)$$

Reflexivity
Example

Example

Recall the following relations; which is reflexive?

$$R_1 = \{(a,b) \mid a \leq b\}$$
$$R_2 = \{(a,b) \mid a,b \in \mathbb{N}, \tfrac{a}{b} \in \mathbb{Z}\}$$
$$R_3 = \{(a,b) \mid a,b \in \mathbb{N}, a-b=2\}$$

- R_1 is reflexive since for every $a \in \mathbb{N}$, $a \leq a$.
- R_2 is not reflexive since $\frac{0}{0}$ is undefined (not an integer). Though all other elements are reflexive.
- R_3 is *not* reflexive since $a - a = 0$ for every $a \in \mathbb{N}$.

Symmetry I
Definition

Definition

A relation R on a set A is called *symmetric* if

$$(b,a) \in R \iff (a,b) \in R$$

for all $a, b \in A$.

A relation R on a set A is called *antisymmetric* if

$$\forall a, b, \left[\big((a,b) \in R \land (b,a) \in R\big) \rightarrow a = b\right]$$

for all $a, b \in A$.

Symmetry II
Definition

Some things to note:

- A symmetric relationship is one in which if a is related to b then b *must* be related to a.
- An antisymmetric relationship is similar, but such relations hold only when $a = b$.
- An antisymmetric relationship is *not necessarily* a reflexive relationship.
- A relation can be both symmetric and antisymmetric or neither or have one property but not the other!
- A relation that is not symmetric is *not asymmetric*.

Relations
Exercises

Prove or disprove the following:

- If a relation R on a set A is reflexive, then it is also symmetric
- If a relation R on a set A is symmetric then it is also reflexive

Provide examples of relations that are:

- Antisymmetric and symmetric
- Antisymmetric but not symmetric
- Not antisymmetric but symmetric
- Neither antisymmetric nor symmetric

Symmetric Relations
Example

Example

Let $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Is R reflexive? Symmetric? Antisymmetric?

- It is clearly not reflexive since for example $(2,2) \notin \mathbb{R}$.
- It is symmetric since $x^2 + y^2 = y^2 + x^2$ (i.e. addition is commutative).
- It is not antisymmetric since $(\frac{1}{3}, \frac{\sqrt{8}}{3}) \in R$ and $(\frac{\sqrt{8}}{3}, \frac{1}{3}) \in R$ but $\frac{1}{3} \neq \frac{\sqrt{8}}{3}$

Transitivity
Definition

Definition

A relation R on a set A is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a,b,c \in R$. Equivalently,

$$\forall a,b,c \in A\big((aRb \land bRc) \rightarrow aRc\big)$$

Transitivity
Examples

Example

Is the relation $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$ transitive?

Yes it is transitive since $(x \leq y) \wedge (y \leq z) \Rightarrow x \leq z$.

Example

Is the relation $R = \{(a, b), (b, a), (a, a)\}$ transitive?

No since bRa and aRb but $b \not R b$.

Transitivity
Examples

Example

Is the relation

$$\{(a, b) \mid a \text{ is an ancestor of } b\}$$

transitive?

Yes, if a is an ancestor of b and b is an ancestor of c then a is also an ancestor of c (who is the youngest here?).

Example

Is the relation $\{(x, y) \mid x^2 \geq y\}$ transitive?

No, for example, $(2, 4) \in R$ and $(4, 10) \in R$ (i.e. $2^2 \geq 4$ and $4^2 = 16 \geq 10$ but $2^2 < 10$ thus $(2, 10) \notin R$.

Other Properties

- A relation is *asymmetric* if

$$\forall a, b\big[(a, b) \in R \rightarrow (b, a) \notin R\big]$$

- A relation is *irreflexive* if

$$\forall a\big[(a, a) \notin R\big]$$

Lemma

A relation R on a set A is asymmetric if and only if R is irreflexive and antisymmetric.

Combining Relations

Relations are simply sets, that is subsets of ordered pairs of the Cartesian product of a set.

It therefore makes sense to use the usual set operations, intersection $\cap$, union $\cup$ and set difference $A \setminus B$ to *combine* relations to create new relations.

Sometimes combining relations endows them with the properties previously discussed. For example, two relations may not be transitive alone, but their union may be.

Combining Relations

Example

Let

$$\begin{aligned} A &= \{1, 2, 3, 4\} \\ B &= \{1, 2, 3\} \\ R_1 &= \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\} \\ R_2 &= \{(1, 1), (1, 2), (1, 3), (2, 3)\} \end{aligned}$$

Then

- $R_1 \cup R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (3, 4), (4, 1), (4, 2)\}$
- $R_1 \cap R_2 = \{(1, 2), (1, 3)\}$
- $R_1 \setminus R_2 = \{(1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$
- $R_2 \setminus R_1 = \{(1, 1), (2, 3)\}$

Relation Composition

Definition

Let R_1 be a relation from the set A to B and R_2 be a relation from B to C. I.e. $R_1 \subseteq A \times B, R_2 \subseteq B \times C$. The *composite* of R and S is the relation consisting of ordered pairs (a, c) where $a \in A, c \in C$ and for which there exists and element $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$. We denote the composite of R_1 and R_2 by

$$R_1 \circ R_2$$

Relation Composition I

Example

Construct $R_1 \circ R_1$:

- $(a_1, a_1) \in R_1 \circ R_1$? (yes, with $b = a_4$)
- $(a_1, a_4) \in R_1 \circ R_1$? (yes, with $b = a_3$)
- $(a_2, a_1) \in R_1 \circ R_1$? (no)

Relation Composition II

Example

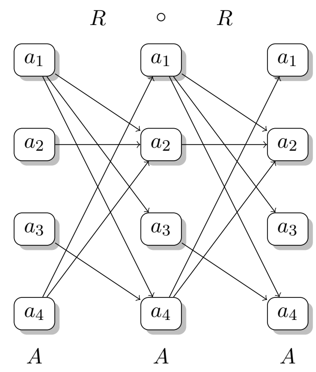


Figure : Composition of a relation with itself.

Powers of Relations

Using this *composite* way of combining relations (similar to function composition) allows us to recursively define *powers* of a relation R.

Definition

Let R be a relation on A. The powers, $R^n, n = 1, 2, 3, \ldots$, are defined recursively by

$$\begin{aligned} R^1 &= R \\ R^{n+1} &= R^n \circ R \end{aligned}$$

Powers of Relations

Example

Powers of Relations

The powers of relations give us a nice characterization of transitivity.

Theorem

A relation R is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Representing Relations

We have seen ways of graphically representing a function/relation between two (different) sets—specifically a graph with arrows between nodes that are related.

We will look at two alternative ways of representing relations; 0-1 matrices and directed graphs.

0-1 Matrices I

A 0-1 matrix is a matrix whose entries are either 0 or 1.

Let R be a relation from $A = \{a_1, a_2, \ldots, a_n\}$ to $B = \{b_1, b_2, \ldots, b_m\}$.

Note that we have induced an ordering on the elements in each set. Though this ordering is arbitrary, it is important to be consistent; that is, once we fix an ordering, we stick with it.

In the case that $A = B$, R is a relation *on* A, and we choose the same ordering.

0-1 Matrices II

The relation R can therefore be represented by a $(n \times m)$ sized 0-1 matrix $\mathbf{M}_R = [m_{i,j}]$ as follows.

$$m_{i,j} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Intuitively, the (i, j)-th entry is 1 if and only if $a_i \in A$ is related to $b_j \in B$.

0-1 Matrices III

An important note: the choice of row or column-major form is important. The (i, j)-th entry refers to the i-th *row* and j-th *column*. The size, $(n \times m)$ refers to the fact that $\mathbf{M}_R$ has n *rows* and m *columns*.

Though the choice is arbitrary, switching between row-major and column-major is a bad idea, since for $A \neq B$, the Cartesian products $A \times B$ and $B \times A$ are not the same.

In matrix terms, the *transpose*, $(\mathbf{M}_R)^T$ does not give the same relation. This point is moot for $A = B$.

0-1 Matrices IV

$$A \left\{ \begin{matrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{matrix} \right. \overbrace{\begin{matrix} b_1 & b_2 & b_3 & b_4 \end{matrix}}^{B} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Let's take a quick look at the example from before.

Matrix Representation

Example

Example

Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$ let R be a relation from A to B as follows:

$$\begin{aligned} R &= \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), \\ &\quad (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_5, b_1)\} \end{aligned}$$

What is $\mathbf{M}_R$?

Clearly, we have a (5×3) sized matrix.

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Matrix Representations

Useful Characteristics

A 0-1 matrix representation makes checking whether a relation is reflexive, symmetric and transitive or not very easy.

Reflexivity – For R to be reflexive, $\forall a(a, a) \in R$. By the definition of the 0-1 matrix, R is reflexive if and only if $m_{i,i} = 1$ for $i = 1, 2, \ldots, n$. Thus, one simply has to check the diagonal.

Matrix Representations
Useful Characteristics

Symmetry – R is symmetric if and only if for all pairs (a, b), $aRb \Rightarrow bRa$. In our defined matrix, this is equivalent to $m_{i,j} = m_{j,i}$ for every pair $i, j = 1, 2, \ldots, n$.

Alternatively, R is symmetric if and only if $\mathbf{M}_R = (\mathbf{M}_R)^T$.

Antisymmetry – To check antisymmetry, you can use a disjunction; that is R is antisymmetric if $m_{i,j} = 1$ with $i \neq j$ then $m_{j,i} = 0$. Thus, for all $i, j = 1, 2, \ldots, n$, $i \neq j$, $(m_{i,j} = 0) \vee (m_{j,i} = 0)$.

What is a simpler logical equivalence?

$$\forall i, j = 1, 2, \ldots, n; i \neq j \big(\neg(m_{i,j} \wedge m_{j,i})\big)$$

Matrix Representations
Example

Example

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Is R reflexive? Symmetric? Antisymmetric?

- Clearly it is not reflexive since $m_{2,2} = 0$.
- It is not symmetric either since $m_{2,1} \neq m_{1,2}$.
- It is, however, antisymmetric. You can verify this for yourself.

Matrix Representations
Combining Relations

Combining relations is also simple—union and intersection of relations is nothing more than entry-wise boolean operations.

Union – An entry in the matrix of the union of two relations $R_1 \cup R_2$ is 1 if and only if at least one of the corresponding entries in R_1 or R_2 is one. Thus

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2}$$

Intersection – An entry in the matrix of the intersection of two relations $R_1 \cap R_2$ is 1 if and only if *both* of the corresponding entries in R_1 and R_2 is one. Thus

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$$

Matrix Representations
Combining Relations

Example

Let

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

What is $\mathbf{M}_{R_1 \cup R_2}$ and $\mathbf{M}_{R_1 \cap R_2}$

$$\mathbf{M}_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \mathbf{M}_{R_1 \cap R_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

How does combining the relations change their properties?

Matrix Representations
Composite Relations

One can also compose relations easily with 0-1 matrices. We will not discuss how here, rather please read this for yourself.

You will need to read section 2.7 for some definitions (Boolean product of matrices).

Remember that recursively composing a relation $R^n, n = 1, 2, \ldots$ gives a nice characterization of transitivity.

Using these ideas, you can also determine if a relation is transitive or not by computing the *transitive closure* (discussed in the next section).

Directed Graphs

We will get more into graphs later on, but we briefly introduce them here since they can be used to represent relations.

In the general case, we've already seen directed graphs used to represent relations. However, for relations *on* a set A, it makes more sense to use a general graph rather than have *two* copies of the set in the diagram.

Directed Graphs I

Definition

A *directed graph* (or *digraph*) consists of a set V of *vertices* (or *nodes*) together with a set E of edges of ordered pairs of elements of V. We write $G = (V, E)$.

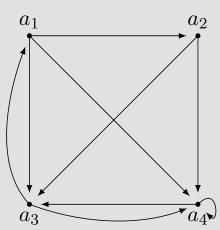
Directed Graphs II

Example

Directed Graphs III

Let $A = \{a_1, a_2, a_3, a_4\}$ and let R be a relation on A defined as:

$$R = \{(a_1,a_2),(a_1,a_3),(a_1,a_4),(a_2,a_3),(a_2,a_4)\\ (a_3,a_1),(a_3,a_4),(a_4,a_3),(a_4,a_4)\}$$



Directed Graph Representation I

Usefulness

Again, a directed graph offers some insight as to the properties of a relation.

Reflexivity – In a digraph, a relation is reflexive if and only if every vertex has a self loop.

Symmetry – In a digraph, a represented relation is symmetric if and only if for every edge from x to y there is also a corresponding edge from y to x.

Directed Graph Representation II

Usefulness

Antisymmetry – A represented relation is antisymmetric if and only if there is never a back edge for each directed edge between distinct vertices.

Transitivity – A digraph is transitive if for every pair of edges (x, y) and (y, z) there is also a directed edge (x, z) (though this may be harder to verify in more complex graphs visually).

Closures

Definition

If a given relation R is not reflexive (or symmetric, antisymmetric, transitive) can we transform it into a relation R' that is?

Example

Let $R = \{(1,2),(2,1),(2,2),(3,1),(3,3)\}$ is not reflexive. How can we make it reflexive?

In general, we'd like to change the relation as *little as possible*. To make this relation reflexive we simply have to add $(1, 1)$ to the set.

Inducing a property on a relation is called its *closure*. In the example, R' is the *reflexive closure*.

Closures I

In general, the reflexive closure of a relation R on A is $R \cup \Delta$ where

$$\Delta = \{(a, a) \mid a \in A\}$$

is the *diagonal relation* on A.

Question: How can we compute the reflexive closure using a 0-1 matrix representation? Digraph representation?

Similarly, we can create symmetric closures using the inverse of a relation. That is, $R \cup R^{-1}$ where

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

Question: How can we compute the symmetric closure using a 0-1 matrix representation? Digraph representation?

Closures II

Also, transitive closures can be made using a previous theorem:

Theorem

A relation R is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Thus, if we can compute R^k such that $R^k \subseteq R^n$ for all $n \geq k$, then R^k is the transitive closure.

To see how to efficiently do this, we present *Warhsall's Algorithm*.

Note: your book gives much greater details in terms of graphs and *connectivity relations*. It is good to read these, but they are predicated on material that we have not yet seen.

Warshall's Algorithm I

Key Ideas

In any set A with $|A| = n$ elements, any transitive relation will be built from a sequence of relations that has a length at most n. Why? Consider the case where A contains the relations

$$(a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n)$$

Then (a_1, a_n) is required to be in A for A to be transitive.

Thus, by the previous theorem, it suffices to compute (*at most*) R^n. Recall that $R^k = R \circ R^{k-1}$ is calculated using a Boolean matrix product. This gives rise to a natural algorithm.

Warshall's Algorithm II

Key Ideas

To put it another way, each 0-1 matrix corresponds to a sequence of relations between a_i and a_j via a relation with some element in A numbered k or less $(a_1, \ldots, a_k)$.

For example $R^{(1)}$ is the matrix of pairs related via a_1

The idea to Warshall's is that we can use $R^{(k-1)}$ to compute $R^{(k)}$ by realizing that there are two situations. Consider a_{ij} in $R^{(k)}$. If $a_{ij} = 1$ then that means there exists a relation from a_i to a_j through a sequence of elements $I \subseteq A$ where each $a_l \in A$ satisfies $l \leq k$.

Running through all possible $k = 1, 2, \ldots, n$ for every pair $1 \leq i, j \leq n$ will compute our transitive closure.

Warshall's Algorithm

WARSHALL'S ALGORITHM

INPUT : An $(n \times n)$ 0-1 Matrix $\mathbf{M}_R$ representing a relation R
OUTPUT : A $(n \times n)$ 0-1 Matrix $\mathbf{W}$ representing the transitive closure of R

```
W = M_R
FOR k = 1, ..., n DO
    FOR i = 1, ..., n DO
        FOR j = 1, ..., n DO
            w_{i,j} = w_{i,j} ∨ (w_{i,k} ∧ w_{k,j})
        END
    END
END
return W
```

Warshall's Algorithm

Example

Example

Compute the transitive closure of the relation

$$R = \{(1,1), (1,2), (1,4), (2,2), (2,3), (3,1), (3,4), (4,1), (4,4)\}$$

on $A = \{1, 2, 3, 4\}$

Equivalence Relations

Consider the set of every person in the world. Now consider a relation such that $(a, b) \in R$ if a and b are siblings.

Clearly, this relation is reflexive, symmetric and transitive. Such a unique relation is called and *equivalence relation*.

Definition

A relation on a set A is an *equivalence relation* if it is reflexive, symmetric and transitive.

Equivalence Classes I

Though a relation on a set A may not be an equivalence relation, we *can* defined a subset of A such that R *does* become an equivalence relation (for that subset).

Definition

Let R be an equivalence relation on the set A and let $a \in A$. The set of all elements in A that are related to a is called the *equivalence class* of a. We denote this set $[a]_R$ (we omit R when there is no ambiguity as to the relation). That is,

$$[a]_R = \{s \mid (a, s) \in R, s \in A\}$$

Equivalence Classes II

Elements in $[a]_R$ are called *representatives* of the equivalence class.

Theorem

Let R be an equivalence relation on a set A. TFAE:

1. aRb
2. $[a] = [b]$
3. $[a] \cap [b] \neq \emptyset$

Partitions I

Equivalence classes are important because they can *partition* a set A into disjoint non-empty subsets $A_1, A_2, \ldots, A_l$ where each equivalence class is, in some sense, self-contained.

Note that a partition satisfies these properties:

- $\bigcup_{i=1}^{l} A_i = A$
- $A_i \cap A_j = \emptyset$ for $i \neq j$
- $A_i \neq \emptyset$ for all i

Partitions II

For example, if R is a relation such that $(a, b) \in R$ if a and b live in the same US state (or outside the US), then R is an equivalence relation that *partitions* US residents into 50 equivalence classes.

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition A_i of the set S, there is an equivalence relation R that has the sets A_i as its equivalence classes.

Visual Interpretation

In a 0-1 matrix, if the elements are ordered into their equivalence classes, equivalence classes/partitions form perfect squares of 1s (and zeros else where).

In a digraph, equivalence classes form a collection of disjoint *complete* graphs.

Example

Say that we have $A = \{1, 2, 3, 4, 5, 6, 7\}$ and R is an equivalence relation that partitions A into $A_1 = \{1, 2\}, A_2 = \{3, 4, 5, 6\}$ and $A_3 = \{7\}$. What does the 0-1 matrix look like? Digraph?

Equivalence Relations

Example I

Example

Let $R = \{(a,b) \mid a,b \in \mathbb{R}, a \leq b\}$

- Reflexive?
- Transitive?
- Symmetric? No, it is not since, in particular $4 \leq 5$ but $5 \not\leq 4$.
- Thus, R is not an equivalence relation.

Equivalence Relations

Example II

Example

Let $R = \{(a,b) \mid a,b \in \mathbb{Z}, a = b\}$

- Reflexive?
- Transitive?
- Symmetric?
- What are the equivalence classes that partition $\mathbb{Z}$?

Equivalence Relations

Example III

Example

For $(x,y),(u,v) \in \mathbb{R}^2$ define

$$R = \{((x,y),(u,v)) \mid x^2 + y^2 = u^2 + v^2\}$$

Show that R is an equivalence relation. What are the equivalence classes it defines (i.e. what are the partitions of $\mathbb{R}$?

Equivalence Relations

Example IV

Example

Given $n, r \in \mathbb{N}$, define the set

$$n\mathbb{Z} + r = \{na + r \mid a \in \mathbb{Z}\}$$

- For $n = 2, r = 0$, $2\mathbb{Z}$ represents the equivalence class of all even integers.
- What n, r give the equivalence class of all *odd* integers?
- If we set $n = 3, r = 0$ we get the equivalence class of all integers divisible by 3.
- If we set $n = 3, r = 1$ we get the equivalence class of all integers divisible by 3 with a *remainder* of one.
- In general, this relation defines equivalence classes that are, in fact, *congruence classes*. (see chapter 2, to be covered later).