Predicate Logic and Quantifiers

Computer Science & Engineering 235

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Introduction

Consider the following statements:

$$x > 3, \quad x = y + 3, \quad x + y = z$$

The truth value of these statements has no meaning without specifying the values of x, y, z.

However, we *can* make propositions out of such statements.

A *predicate* is a property that is affirmed or denied about the *subject* (in logic, we say "variable" or "argument") of a *statement*.

$$\text{``}\underbrace{x}_{\text{subject}} \underbrace{\text{is greater than 3''}}_{\text{predicate}}$$

Propositional Functions

Definition

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the *propositional function* P. Here, $(x_1, x_2, \ldots, x_n)$ is an n-tuple and P is a predicate.

You can think of a propositional function as a function that

- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.

Propositional Functions

Example

Example

Let $Q(x, y, z)$ denote the statement "$x^2 + y^2 = z^2$". What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of (x, y, z) make the predicate true?

Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.

Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for (x, y, z) that make this propositional function true—how many right triangles are there?

Universe of Discourse

Consider the previous example. Does it make sense to assign to x the value "blue"?

Intuitively, the *universe of discourse* is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function $P(x) =$ "The test will be on x the 23rd" be?

Universe of Discourse

Multivariate Functions

Moreover, each variable in an n-tuple may have a different universe of discourse.

Let $P(r, g, b, c) =$ "The rgb-value of the color c is (r, g, b)".

For example, $P(255, 0, 0, \textit{red})$ is true, while $P(0, 0, 255, \textit{green})$ is false.

What are the universes of discourse for (r, g, b, c)?

Quantifiers
Introduction

A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to *quantify* it. That is, the predicate is true (or false) for *all* possible values in the universe of discourse or for *some* value(s) in the universe of discourse.

Such *quantification* can be done with two *quantifiers*: the *universal* quantifier and the *existential* quantifier.

Universal Quantifier
Definition

Definition

The *universal quantification* of a predicate $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of all elements:

$$\forall x P(x) \iff P(n_1) \wedge P(n_2) \wedge \cdots \wedge P(n_k)$$

Universal Quantifier
Example I

- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".

$$\forall x (Q(x) \rightarrow P(x))$$

- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

$$\forall x (Q(x) \vee P(x))$$

- Are these statements true or false?

Universal Quantifier
Example II

Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Note that we can also use the shorthand

$$\forall x, y P(x, y)$$

Existential Quantifier
Definition

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Again, if the universe of discourse is finite, $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of all elements:

$$\exists x P(x) \iff P(n_1) \vee P(n_2) \vee \cdots \vee P(n_k)$$

Existential Quantifier
Example I

Let $P(x, y)$ denote the statement, "$x + y = 5$".

What does the expression,

$$\exists x \exists y P(x, y)$$

mean?

What universe(s) of discourse make it true?

Existential Quantifier
Example II

Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

The statement can thus be expressed as

$$\exists x P(x)$$

Existential Quantifier
Example II Continued

Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such *real* number x can satisfy the predicate.

How can we make it so that it *is* true?

Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$.

Quantifiers
Truth Values

In general, when are quantified statements true/false?

Statement	True When	False When
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Table : Truth Values of Quantifiers

Mixing Quantifiers I

Existential and universal quantifiers can be used together to quantify a predicate statement; for example,

$$\exists x \forall y P(x, y)$$

is perfectly valid. However, you must be careful—it must be read left to right.

You *can* commute *similar* quantifiers:

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

Which is why our shorthand was valid.

Mixing Quantifiers II

However, dissimilar quantifiers do *not* commute; in general,

$$\exists x \forall y P(x, y) \not\equiv \forall y \exists x P(x, y)$$

Consider as an example the additive inverse law for integers: for every integer x, there is an integer y such that $x + y = 0$; with $P(x, y) : x + y = 0$, we can define

$$\forall x \exists y P(x, y)$$

which holds because for each integer x, its additive inverse is $-x$. The additive inverse for each integer is *unique* to that integer. Now consider the statement with commuted quantifiers:

$$\exists y \forall x P(x, y)$$

There does not exist a *single* integer y that acts as an additive inverse for all other integers x.

Mixing Quantifiers
Truth Values

Statement	True When	False When
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair, x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table : Truth Values of 2-variate Quantifiers

Mixing Quantifiers
Example I

Express, in predicate logic, the axiom that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following.

$$\forall x \exists y P(x, y)$$

Mixing Quantifiers
Example II: More Mathematical Axioms

Express the *commutative law of addition* for $\mathbb{R}$.

We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Then we have the following:

$$\forall x \forall y (x + y = y + x)$$

Mixing Quantifiers
Example II: More Mathematical Axioms Continued

Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

We want to express that for every real number x, there exists a real number y such that $xy = 1$.

Then we have the following:

$$\forall x \exists y (xy = 1)$$

Mixing Quantifiers
Example II: False Mathematical Axioms

Is commutativity for subtraction valid over the reals?

That is, for all pairs of real numbers x, y does the identity $x - y = y - x$ hold? Express this using quantifiers.

The expression is

$$\forall x \forall y (x - y = y - x)$$

This is clearly false: $x = 2, y = 3$ so the negation is true:

$$\neg \forall x, y [x - y = y - x] \equiv \exists x, y [x - y \neq y - x]$$

Negation

Just as we can use negation with propositions, we can use them with quantified expressions.

Lemma

Let $P(x)$ be a predicate. Then the following hold.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).

Negation
Truth Values

Statement	True When	False When
$\neg \exists x P(x) \equiv \forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x P(x) \equiv \exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Table : Truth Values of Negated Quantifiers

Mixing Quantifiers

Example II: False Mathematical Axioms Continued

Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer x does there exists a y such that $xy = 1$?

This is false, since we can find a *counter example*. Take any integer, say 5 and multiply it with another integer, y. If the axiom held, then $5 = 1/y$, but for any (nonzero) integer y, $|1/y| \leq 1$.

Mixing Quantifiers

Exercise

> *Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.*

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".
- Then the expression is

$$\exists x \forall y \left(P(x, y) \wedge Q(x, y) \right)$$

- Over what universe(s) of discourse does this axiom hold?
- This is the *additive identity law* and holds for $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$ but does not hold for $\mathbb{Z}^+$.

Distributivity with Quantifiers

- Quantifiers can be distributed *in certain cases*
- Universal quantifiers distribute over conjunctions:

$$\forall x \left[P(x) \wedge Q(x) \right] \equiv \forall x P(x) \wedge \forall x Q(x)$$

- Existential quantifiers distribute over disjnctions

$$\exists x \left[P(x) \vee Q(x) \right] \equiv \exists x P(x) \vee \exists x Q(x)$$

- Universal quantifiers *do not* distribute over disjunctions:

$$\forall x \left[P(x) \vee Q(x) \right] \not\equiv \forall x P(x) \vee \forall x Q(x)$$

- Existential quantifiers *do not* distribute over conjunctions:

$$\exists x \left[P(x) \wedge Q(x) \right] \not\equiv \exists x P(x) \wedge \exists x Q(x)$$

Separating Mixed quantifies do not commute

- Does the following equivalence hold?

$$\forall x, y P(x, y) \equiv \forall x P(x, y) \wedge \forall y P(x, y)$$

- The above is *false*
- Left-hand side: both variables are bound
- Right-hand side, first expression: x is bound, y is free
- Second expression: y is bound, x is free
- *All variables that occur in a propositional function must be bound to turn it into a proposition.*
- The left-hand side is a proposition, but the right-hand side is not

Binding Variables I

When a quantifier is used on a variable x, we say that x is *bound*. If no quantifier is used on a variable in a predicate statement, it is called *free*.

Example

In the expression $\exists x \forall y P(x, y)$ both x and y are bound.

In the expression $\forall x P(x, y)$, x is bound, but y is free.

Binding Variables II

The set of all variables bound by a common quantifier is the *scope* of that quantifier.

Example

In the expression $\exists x, y \forall z P(x, y, z, c)$ the scope of the existential quantifier is $\{x, y\}$, the scope of the universal quantifier is just z and c has no scope since it is free.

Vacuousness

For any universe of discourse that is *empty*, then for any predicate,

$$\exists x P(x)$$

is always false, and

$$\forall x P(x)$$

is always true. The second is always *vacuously* true ($P(x)$ holds for every x in the universe of discourse because there is no such x).

Conclusion

Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y\forall z P(x,y,z) \wedge \exists z\forall y P(x,y,z)\big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x\big(\forall y\exists z\neg P(x,y,z) \vee \forall z\exists y\neg P(x,y,z)\big)$$

- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \to ((x=y) \vee (x=1))\big]$". When is $Q(y)$ true?
- Answer: Only when y is a prime number.