Partial Orders

Computer Science & Engineering 235: Discrete Mathematics

Christopher M. Bourke
cbourke@cse.unl.edu

Partial Orders I

Motivating Introduction

Consider a renovation of an older building. In this process several things need to be done.

- Remove Asbestos
- Replace Windows
- Paint Walls
- Refinish Floors
- Assigning Offices
- Moving in Office Furniture.

Partial Orders II

Motivating Introduction

- Some things have to be done before others can begin
- Example: Asbestos has to be removed before *anything*; painting has to be done before the floors to avoid ruining them, etc.
- Conversely: several things can be been done concurrently
- Example: painting could be done while replacing the windows and assigning office could be been done at anytime

Such a scenario can be nicely modeled using *partial orderings*.

Partial Orderings I

Definition

Definition

A relation R on a set S is called a *partial order* if it is reflexive, antisymmetric and transitive. A set S together with a partial ordering R is called a *partially ordered set* or *poset* for short and is denoted

$$(S, R)$$

Partial orderings are used to give an order to sets that may not have a natural one. In our renovation example, we could define an ordering such that $(a, b) \in R$ if a *must* be done before b can be done.

Partial Orderings II

Definition

We use the notation

$$a \preccurlyeq b$$

to indicate that $(a, b) \in R$ is a partial order and

$$a \prec b$$

when $a \neq b$.

The notation $\prec$ is not to be mistaken for "less than equal to." Rather, $\prec$ is used to denote *any* partial ordering.

Partial Orderings

Comparability

Definition

The elements a and b of a poset $(S, \prec)$ are called comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When $a, b \in S$ such that neither are comparable, we say that they are *incomparable*.

Looking back at our renovation example, we can see that

$$\text{Remove Asbestos} \prec a_i$$

for all activities a_i. Also,

$$\text{Paint Walls} \prec \text{Refinish Floors}$$

Some items are also incomparable—replacing windows can be done before, after or during the assignment of offices.

Partial Orderings

Total Orders

Definition

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered set*. The relation $\preccurlyeq$ is said to be a *total order*.

Example

The set of integers over the relation "less than equal to" is a total order; $(\mathbb{Z}, \leq)$ since for every $a, b \in \mathbb{Z}$, it must be the case that $a \leq b$ or $b \leq a$.

What happens if we replace $\leq$ with $<$?

Partial Orderings

Well-Orderings

Definition

$(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and such that every nonempty subset of S has a *least element*

Example

The natural numbers along with $\leq$, $(\mathbb{N}, \leq$ is a well-ordered set since any subset of $\mathbb{N}$ will have a least element and $\leq$ is a total ordering on $\mathbb{N}$ as before.

Principle of Well-Ordered Induction

Well-ordered sets are the basis of the proof technique known as *induction* (more when we cover Chapter 3).

Theorem (Principle of Well-Ordered Induction)

Suppose that S is a well ordered set. Then $P(x)$ is true for all $x \in S$ if

Basis Step: *$P(x_0)$ is true for the least element of S and*
Induction Step: *For every $y \in S$ if $P(x)$ is true for all $x \prec y$ then $P(y)$ is true.*

More examples??

Lexicographic Orderings I

Lexicographic ordering is the same as any dictionary or phone book—we use alphabetical order starting with the first character in the string, then the next character (if the first was equal) etc. (you can consider "no character" for shorter words to be less than "a").

Lexicographic Orderings II

Formally, lexicographic ordering is defined by combining two other orderings.

Definition

Let $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$ be two posets. The *lexicographic ordering* $\preccurlyeq$ on the Cartesian product $A_1 \times A_2$ is defined by

$$(a_1, a_2) \preccurlyeq (a'_1, a'_2)$$

if $a_1 \prec_1 a'_1$ or if $a_1 = a'_1$ and $a_2 \preccurlyeq_2 a'_2$.

Lexicographic Orderings III

Lexicographic ordering generalizes to the Cartesian product of n sets in the natural way.

Define $\preccurlyeq$ on $A_1 \times A_2 \times \cdots \times A_n$ by

$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$$

if $a_1 \prec b_1$ or if there is an integer $i > 0$ such that

$$a_1 = b_1, a_2 = b_2, \ldots, a_i = b_i$$

and $a_{i+1} \prec b_{i+1}$

Hasse Diagrams

As with relations and functions, there is a convenient graphical representation for partial orders—*Hasse Diagrams*.

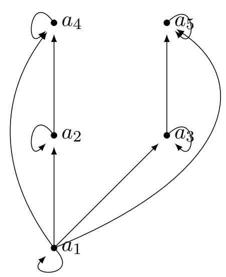
Consider the digraph representation of a partial order—since we *know* we are dealing with a partial order, we *implicitly* know that the relation must be reflexive and transitive. Thus we can simplify the graph as follows:

- Remove all self-loops.
- Remove all transitive edges.
- Make the graph direction-less—that is, we can assume that the orientations are *upwards*.

The resulting diagram is far simpler.

Hasse Diagram

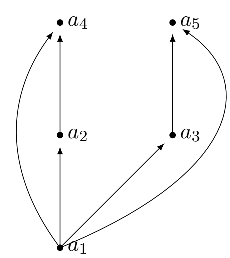
Example



Remove Self-Loops

Hasse Diagram

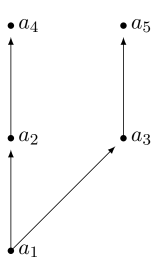
Example



Remove Transitive Loops

Hasse Diagram

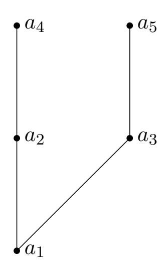
Example



Remove Orientation

Hasse Diagram

Example



Hasse Diagram!

Hasse Diagrams
Example

Of course, you need not always start with the complete relation in the partial order and then trim everything. Rather, you can build a Hasse directly from the partial order.

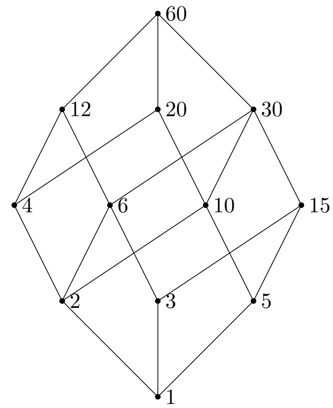
Example

Draw a Hasse diagram for the partial ordering

$$\{(a, b) \mid a \mid b\}$$

on $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$ (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)

Hasse Diagrams
Example Answer



Partial Order Proof
Example

Let's prove that $a|b$ defines a poset.

$a|b$ is reflexive. To see this, let $k = 1$ then

$$a = a \cdot k$$

and so $a|a$ for all integers $a > 1$.

$a|b$ is antisymmetric. Suppose that $a|b$ and $b|a$. Then

$$b = ak_1 \ \wedge a = bk_2$$

Solving one in terms of the other gives us

$$b = bk_1k_2$$

This leaves only two possibilities:

$$k_1 = k_2 = 1$$

or

$$k_1 = k_2 = -1$$

In the first case, plugging it back into the definitions, we get $a = b$.

Extremal Elements I

Definition

An element a in a poset $(S, \preccurlyeq)$ is called *maximal* if it is not less than any other element in S. That is,

$$\nexists b \in S(a \prec b)$$

If there is one *unique* maximal element a, we call it the *maximum* element (or the *greatest element*).

Extremal Elements II

Definition

An element a in a poset $(S, \preccurlyeq)$ is called *minimal* if it is not greater than any other element in S. That is,

$$\nexists b \in S(b \prec a)$$

If there is one *unique* minimal element a, we call it the *minimum* element (or the *least element*).

Extremal Elements III

Definition

Let $(S, \preccurlyeq)$ be a poset and let $A \subseteq S$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$ then u is an *upper bound* of A.

An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the *least upper bound* on A. We abbreviate "lub".

Extremal Elements IV

Definition

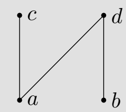
Let $(S, \preccurlyeq)$ be a poset and let $A \subseteq S$. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$ then l is a *lower bound* of A.

An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the *greatest lower bound* on A. We abbreviate "glb".

Extremal Elements

Example I

Example



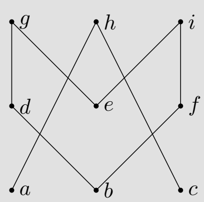
What are the minimal, maximal, minimum, maximum elements?

- Minimal: $\{a, b\}$
- Maximal: $\{c, d\}$
- There are no unique minimal or maximal elements.

Extremal Elements

Example II

Example



What are the lower/upper bounds and glb/lub of the sets $\{d, e, f\}$, $\{a, c\}$ and $\{b, d\}$, $\{b, e\}$

Extremal Elements

Example II

$\{d, e, f\}$

- Lower Bounds: $\emptyset$, thus no glb either.
- Upper Bounds: $\emptyset$, thus no lub either.

$\{a, c\}$

- Lower Bounds: $\emptyset$, thus no glb either.
- Upper Bounds: $\{h\}$, since its unique, lub is also h.

$\{b, d\}$

- Lower Bounds: $\{b\}$ and so also glb.
- Upper Bounds: $\{d, g\}$ and since $d \prec g$, the lub is d.

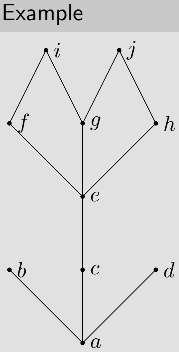
$\{b, e\}$

- Lower Bounds: $\emptyset$
- Upper Bounds: $\{g, i\}$ but neither bounds each other, so no

Extremal Elements

Example III

Example



Minimal/Maximal elements?

- Minimal & Minimum Element: a.
- Maximal Elements: b, d, i, j.

Bounds, glb, lub of $\{c, e\}$?

- Lower Bounds: $\{a, c\}$, thus glb is c.
- Upper Bounds: $\{e, f, g, h, i.j\}$ thus lub is e

Bounds, glb, lub of $\{b, i\}$?

- Lower Bounds: $\{a\}$, thus glb is a.
- Upper Bounds: $\emptyset$, thus lub DNE.

Lattices

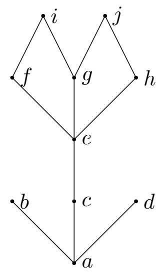
A special structure arises when every pair of elements in a poset has a lub and glb.

Definition

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a *lattice*.

Lattices
Example

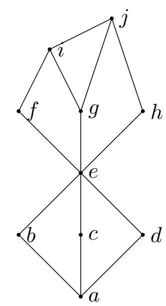
Is the example from before a lattice?



No, since the pair (b, c) do not have a least upper bound.

Lattices
Example

What if we modified it as follows?



Yes, it is now a lattice, since for any pair, there is a lub & glb.

Lattices

To show that a partial order is not a lattice, it suffices to find a pair that does not have a lub/glb.

For a pair not to have a lub/glb, they must first be *incomparable*. (Why?)

You can then view the upper/lower bounds on a pair as a sub-hasse diagram; if there is no *minimum* element in this sub-diagram, then it is not a lattice.

Topological Sorting
Introduction

Let's return to the introductory example of the Avery renovation. Now that we've got a partial order model, it would be nice to actually create a concrete schedule.

That is, given a partial order, we'd like to transform it into a *total order* that is *compatible* with the partial order.

A total order is compatible if it doesn't violate any of the original relations in the partial ordering.

Essentially, we are simply imposing an order on incomparable elements in the partial order.

Preliminaries

Before we give the algorithm, we need some tools to justify its correctness.

Fact

Every finite, nonempty poset $(S, \preccurlyeq)$ *has a minimal element.*

We'll prove by a form of reductio ad absurdum.

Preliminaries
Proof

Proof.

Assume to the contrary that a nonempty, finite (WLOG, assume $|S| = n$) poset $(S \preccurlyeq)$ has no minimal element. In particular, a_1 is not a minimal element.

If a_1 is not minimal, then there exists a_2 such that $a_2 \prec a_1$. But also, a_2 is not minimal by the assumption.

Therefore, there exists a_3 such that $a_3 \prec a_2$. This process proceeds until we have the last element, a_n thus,

$$a_n \prec a_{n-1} \prec \cdots a_2 \prec a_1$$

thus by definition a_n is the minimal element. □

Topological Sorting

Intuition

The idea to topological sorting is that we start with a poset $(S, \preccurlyeq)$ and remove a minimal element (choosing arbitrarily if there are more than one). Such an element is guaranteed to exist by the previous fact.

As we remove each minimal element, the set shrinks. Thus, we are guaranteed the algorithm will halt in a finite number of steps.

Furthermore, the order in which elements are removed is *a* total order;

$$a_1 \prec a_2 \prec \cdots \prec a_n$$

We now present the algorithm itself.

Topological Sorting

Algorithm

TOPOLOGICAL SORT

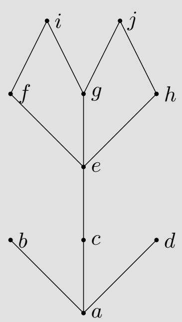
```
INPUT   : (S, ≼) a poset with |S| = n
OUTPUT  : A total ordering (a_1, a_2, ..., a_n)
k = 1
WHILE S ≠ ∅ DO
    a_k ← a minimal element in S
    S = S \ {a_k}
    k = k + 1
END
return (a_1, a_2, ..., a_n)
```

Topological Sorting

Example

Example

Find a compatible ordering (topological ordering) of the poset represented by the diagram below.



Conclusion

Questions? Examples?