Functions

Computer Science & Engineering 235 - Discrete Mathematics

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Introduction

You've already encountered *functions* throughout your education.

$$\begin{aligned} f(x,y) &= x+y \\ f(x) &= x \\ f(x) &= \sin x \end{aligned}$$

Here, however, we will study functions on *discrete* domains and ranges. Moreover, we generalize functions to mappings. Thus, there may not always be a "nice" way of writing functions like above.

Definition

Function

Definition

A *function* f from a set A to a set B is an assignment of exactly one element of B to *each element of* A. We write $f(a) = b$ if b is the *unique* element of B assigned by the function f to the element $a \in A$. If f is a function from A to B, we write

$$f : A \to B$$

This can be read as "f maps A to B".

Note the subtlety:

- Each and every element in A has a *single* mapping.
- Each element in B *may* be mapped to by *several* elements in A or not at all.

Definitions

Terminology

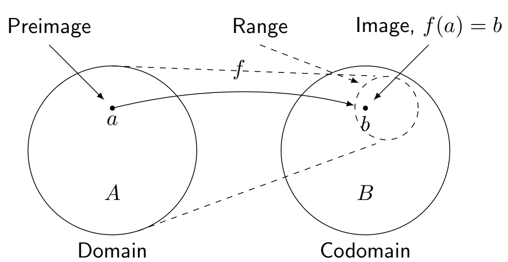
Definition

Let $f : A \to B$ and let $f(a) = b$. Then we use the following terminology:

- A is the *domain* of f, denoted $\mathrm{dom}(f)$.
- B is the *codomain* of f.
- b is the *image* of a.
- a is the *preimage* of b.
- The *range* of f is the set of all images of elements of A, denoted $\mathrm{rng}(f)$.

Definitions

Visualization



A function, $f : A \to B$.

Definition I

More Definitions

Definition

Let f_1 and f_2 be functions from a set A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined by

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ (f_1 f_2)(x) &= f_1(x) f_2(x) \end{aligned}$$

Example

Definition II

More Definitions

Let $f_1(x) = x^4 + 2x^2 + 1$ and $f_2(x) = 2 - x^2$ then

$$\begin{aligned}(f_1 + f_2)(x) &= (x^4 + 2x^2 + 1) + (2 - x^2) \\ &= x^4 + x^2 + 3 \\ (f_1 f_2)(x) &= (x^4 + 2x^2 + 1) \cdot (2 - x^2) \\ &= -x^6 + 3x^2 + 2\end{aligned}$$

Definition

Let $f : A \to B$ and let $S \subseteq A$. The *image* of S is the subset of B that consists of all the images of the elements of S. We denote the image of S by $f(S)$, so that

$$f(S) = \{f(s) \mid s \in S\}$$

Definition III

More Definitions

Note that here, an *image* is a *set* rather than an element.

Example

Let

- $A = \{a_1, a_2, a_3, a_4, a_5\}$
- $B = \{b_1, b_2, b_3, b_4\}$
- $f = \{(a_1, b_2), (a_2, b_3), (a_3, b_3), (a_4, b_1), (a_5, b_4)\}$
- $S = \{a_1, a_3\}$

Draw a diagram for f.

The *image* of S is $f(S) = \{b_2, b_3\}$

Definition

Definition IV

More Definitions

A function f whose domain and codomain are subsets of the set of real numbers is called *strictly increasing* if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f. A function f is called *strictly decreasing* if $f(x) > f(y)$ whenever $x < y$ and x and y are in the domain of f.

Injections, Surjections, Bijections I

Definitions

Definition

A function f is said to be *one-to-one* (or *injective*) if

$$f(x) = f(y) \Rightarrow x = y$$

for all x and y in the domain of f. A function is an *injection* if it is one-to-one.

Intuitively, an injection simply means that each element in A uniquely maps to an element in B.

It may be useful to think of the contrapositive of this definition:

$$x \neq y \Rightarrow f(x) \neq f(y)$$

Injections, Surjections, Bijections II

Definitions

Definition

A function $f : A \to B$ is called *onto* (or *surjective*) if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a *surjection* if it is onto.

Again, intuitively, a surjection means that every element in the codomain is mapped. This implies that the range is the same as the codomain.

Injections, Surjections, Bijections III

Definitions

Definition

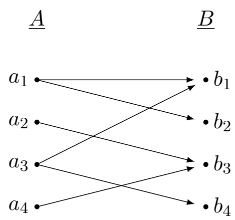
A function f is a *one-to-one correspondence* (or a *bijection*, if it is *both* one-to-one and onto.

One-to-one correspondences are important because they endow a function with an *inverse*. They also allow us to have a concept of cardinality for infinite sets!

A quick note: you may remember the basic "horizontal line test" and "vertical line test", however, these are not necessarily useful on general discrete domains.

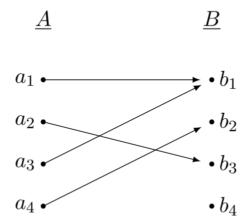
Let's take a look at a few general examples to get the feel for these definitions.

Function Examples
A Non-function



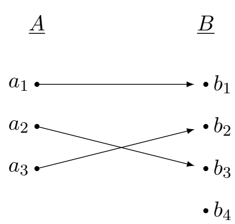
This is not a function: Both a_1 and a_2 map to more than one element in B.

Function Examples
A Function; Neither One-To-One Nor Onto



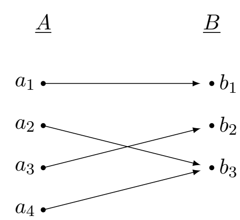
This function not one-to-one since a_1 and a_3 both map to b_1. It is not onto either since b_4 is not mapped to by any element in A.

Function Examples
One-To-One, Not Onto



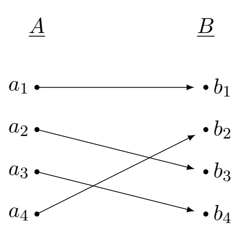
This function is one-to-one since every $a_i \in A$ maps to a unique element in B. However, it is not onto since b_4 is not mapped to by any element in A.

Function Examples
Onto, Not One-To-One



This function is onto since every element $b_i \in B$ is mapped to by some element in A. However, it is not one-to-one since b_3 is mapped to more than one element in A.

Function Examples
A Bijection



This function is a bijection because it is both one-to-one and onto; every element in A maps to a unique element in B and every element in B is mapped by some element in A.

Proving Properties I

Given a mapping $f : A \to B$: how do we tell if:

- f is a function
- f is onto
- f is one-to-one
- f is a bijection

Proving Properties II

Is f a function?

- Does every element in A get mapped to a single element in B?
- Is $f(a) \in B$ for all $a \in A$?
- Test: let $a \in A$, what is $f(a)$?
- Might there be an $a \in A$ such that $f(a) \notin B$? (then *not* a function)

Proving Properties III

Is f onto?

- Let $b \in \text{rng}(f)$
- Can b be generalized to the entire codomain?
- If there is some property of b that does not generalize, then not onto
- Is the range bounded (does it have a global minimum/maximum)?

Proving Properties IV

Is f one-to-one?

- General approach: Let $x_1, x_2 \in \text{dom}(f)$ and $x_1 \neq x_2$
- Assume f is *not* one-to-one;
- Then x_1, x_2 map to the same element, $f(x_1) = f(x_2)$
- Apply the definition of $f(x)$ to see what happens;
- If a contradiction occurs, then the function *is* one-to-one
- Otherwise, might provide insight as to two elements that *do* map to the same element
- If that's the case, then f is *not* one-to-one

Exercises I

Exercise I

Example

Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = 2x - 3$$

What is the domain and range of f? Is it onto? One-to-one?

Clearly, $\text{dom}(f) = \mathbb{Z}$. To see what the range is, note that

$$\begin{aligned} b \in \text{rng}(f) &\iff b = 2a - 3 \qquad a \in \mathbb{Z} \\ &\iff b = 2(a-1) - 1 \\ &\iff b \text{ is odd} \end{aligned}$$

Exercises II

Exercise I

Therefore, the range is the set of all *odd* integers. Since the range and codomain are different, (i.e. $\text{rng}(f) \neq \mathbb{Z}$) we can also conclude that f is *not* onto.

However, f is one-to-one. To prove this, note that

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow 2x_1 - 3 = 2x_2 - 3 \\ &\Rightarrow x_1 = x_2 \end{aligned}$$

follows from simple algebra.

Exercises

Exercise II

Example

Let f be as before,

$$f(x) = 2x - 3$$

but now define $f : \mathbb{N} \to \mathbb{N}$. What is the domain and range of f? Is it onto? One-to-one?

By changing the domain/codomain in this example, f is not even a function anymore. Consider $f(1) = 2 \cdot 1 - 3 = -1 \notin \mathbb{N}$.

Exercises I
Exercise III

Example

Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = x^2 - 5x + 5$$

Is this function one-to-one? Onto?

It is *not* one-to-one since for

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5 \\ &\Rightarrow x_1^2 - 5x_1 = x_2^2 - 5x_2 \\ &\Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2 \\ &\Rightarrow (x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2) \\ &\Rightarrow (x_1 + x_2) = 5 \end{aligned}$$

Exercises II
Exercise III

- Any $x_1, x_2 \in \mathbb{Z}$ that satisfies the equality will map to the same element
- $f(2) = f(3) = -1$
- $f(10) = f(-5) = 55$

f is *not* onto:

- f is a parabola with global minimum (calculus exercise) at $(\frac{5}{2}, -\frac{5}{4})$
- $f(x) = -2$ cannot happen for any $x \in \mathbb{Z}$
- Observe: if $x^2 - 5x + 5 = -2$ then $x = \frac{5 \pm i\sqrt{3}}{2}$ which is not an integer

What would happen if we changed the domain/codomain?

Exercises I
Exercise IV

Example

Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = 2x^2 + 7x$$

Is this function one-to-one? Onto?

Again, since this is a parabola, it cannot be onto (where is the global minimum?).

Exercises II
Exercise IV

However, it *is* one-to-one. We follow a similar argument as before:

- Assume (by way of contradiction) that f *is not* 1-1
- Then $\exists x_1, x_2 \in \mathbb{Z}$ with $x_1 \neq x_2$ and
- $f(x_1) = f(x_2)$

Observe:

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow 2x_1^2 + 7x_1 = 2x_2^2 + 7x_2 \\ &\Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \\ &\Rightarrow (x_1 + x_2) = -\frac{7}{2} \end{aligned}$$

Exercises III
Exercise IV

- However, $-\frac{7}{2} \notin \mathbb{Z}$
- Therefore, it must be the case that $x_1 = x_2$
- It follows that f is one-to-one.

This is a perfect example of why the horizontal line test fails for general functions.

Exercises I
Exercise V

Example

Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) = 3x^3 - x$$

Is f one-to-one? Onto?

To see if its one-to-one, again suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{Z}$. Then

$$\begin{aligned} 3x_1^3 - x_1 = 3x_2^3 - x_2 &\Rightarrow 3(x_1^3 - x_2^3) = (x_1 - x_2) \\ &\Rightarrow 3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = (x_1 - x_2) \\ &\Rightarrow (x_1^2 + x_1x_2 + x_2^2) = \frac{1}{3} \end{aligned}$$

Exercises II

Exercise V

Again, this is impossible since x_1, x_2 are integers, thus f is one-to-one.

However, the function is *not* onto. Consider this counter example: $f(a) = 1$ for some integer a. If this were true, then it must be the case that

$$a(3a^2 - 1) = 1$$

Where a and $(3a^2 - 1)$ are integers. But the only time we can ever get that the product of two integers is 1 is when we have $-1(-1)$ or $1(1)$ neither of which satisfy the equality.

Inverse Functions I

Definition

Let $f : A \to B$ be a bijection. The *inverse function* of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Thus $f^{-1}(b) = a$ when $f(a) = b$.

More succinctly, if an inverse exists,

$$f(a) = b \iff f^{-1}(b) = a$$

Inverse Functions II

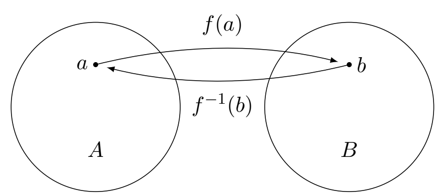
Note that by the definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is *invertible*.

Why must a function be bijective to have an inverse?

- Consider the case where f is not one-to-one. This means that some element $b \in B$ is mapped to by more than one element in A; say a_1 and a_2. How can we define an inverse? Does $f^{-1}(b) = a_1$ or a_2?
- Consider the case where f is not onto. This means that there is some element $b \in B$ that is not mapped to by any $a \in A$, therefore what is $f^{-1}(b)$?

Inverse Functions

Figure



A function & its inverse.

Examples

Example I

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x - 3$$

What is f^{-1}?

First, verify that f is a bijection (it is). To find an inverse, we use substitution:

- Let $f^{-1}(y) = x$
- Let $y = 2x - 3$ and solve for x
- Clearly, $x = \frac{y+3}{2}$ so,
- $f^{-1}(y) = \frac{y+3}{2}$.

Examples

Example II

Example

Let

$$f(x) = x^2$$

What is f^{-1}?

No domain/codomain has been specified. Say $f : \mathbb{R} \to \mathbb{R}$ Is f a bijection? Does an inverse exist?

No, however if we specify that

$$A = \{x \in \mathbb{R} \mid x \leq 0\}$$

and

$$B = \{y \in \mathbb{R} \mid y \geq 0\}$$

then it becomes a bijection and thus has an inverse.

Examples
Example II Continued

To find the inverse, we again, let $f^{-1}(y) = x$ and $y = x^2$. Solving for x we get $x = \pm\sqrt{y}$. But which is it?

Since $\mathrm{dom}(f)$ is all nonpositive and $\mathrm{rng}(f)$ is nonnegative, y must be positive, thus

$$f^{-1}(y) = -\sqrt{y}$$

Thus, it should be clear that domains/codomains are just as important to a function as the definition of the function itself.

Examples
Example III

Example

Let

$$f(x) = 2^x$$

What should the domain/codomain be for this to be a bijection? What is the inverse?

The function should be $f : \mathbb{R} \to \mathbb{R}^+$. What happens when we include 0? Restrict either one to $\mathbb{Z}$?

Let $f^{-1}(y) = x$ and $y = 2^x$, solving for x we get $x = \log_2(x)$.

Therefore,

$$f^{-1}(y) = \log_2(y)$$

Composition I

The values of functions can be used as the input to other functions.

Definition

Let $g : A \to B$ and let $f : B \to C$. The *composition* of the functions f and g is

$$(f \circ g)(x) = f(g(x))$$

Composition II

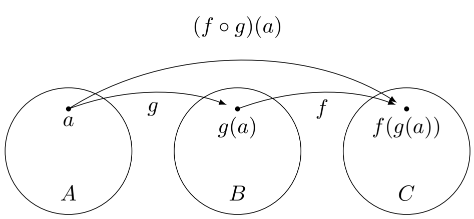
Note the *order* that you apply a function matters—you go from inner most to outer most.

The composition $f \circ g$ cannot be defined unless the the range of g is a subset of the domain of f;

$$f \circ g \text{ is defined} \iff \mathrm{rng}(g) \subseteq \mathrm{dom}(f)$$

It also follows that $f \circ g$ is not necessarily the same as $g \circ f$.

Composition of Functions
Figure



The composition of two functions.

Composition
Example I

Example

Let f and g be functions, $\mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} f(x) &= 2x - 3 \\ g(x) &= x^2 + 1 \end{aligned}$$

What are $f \circ g$ and $g \circ f$?

Note that f is bijective, thus $\mathrm{dom}(f) = \mathrm{rng}(f) = \mathbb{R}$. For g, we have that $\mathrm{dom}(g) = \mathbb{R}$ but that $\mathrm{rng}(g) = \{x \in \mathbb{R} \mid x \geq 1\}$.

Composition

Example I

Even so, $\mathrm{rng}(g) \subseteq \mathrm{dom}(f)$ and so $f \circ g$ is defined. Also, $\mathrm{rng}(f) \subseteq \mathrm{dom}(g)$ so $g \circ f$ is defined as well.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2+1) \\ &= 2(x^2+1)-3 \\ &= 2x^2-1\end{aligned}$$

and

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x-3) \\ &= (2x-3)^2+1 \\ &= 4x^2-12x+10\end{aligned}$$

Equality

Though intuitive, we formally state what it means for two functions to be equal.

Lemma

Two functions f and g are equal if and only if $\mathrm{dom}(f) = \mathrm{dom}(g)$ *and*

$$\forall a \in \mathrm{dom}(f)(f(a) = g(a))$$

Associativity

Though the composition of functions is not commutative ($f \circ g \neq g \circ f$), it *is associative*.

Lemma

Composition of functions is an associative operation; that is,

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Important Functions

Identity Function

Definition

The *identity function* on a set A is the function

$$\iota : A \to A$$

defined by $\iota(a) = a$ for all $a \in A$. This symbol is the Greek letter iota.

One can view the identity function as a composition of a function and its inverse;

$$\iota(a) = (f \circ f^{-1})(a)$$

Moreover, the composition of any function f with the identity function is itself f;

$$(f \circ \iota)(a) = (\iota \circ f)(a) = f(a)$$

Inverses & Identity

The identity function, along with the composition operation gives us another characterization for when a function has an inverse.

Theorem

Functions $f : A \to B$ and $g : B \to A$ are inverses if and only if

$$g \circ f = \iota_A \text{ and } f \circ g = \iota_B$$

That is,

$$\forall a \in A, b \in B\,[(g(f(a)) = a \wedge f(g(b)) = b]$$

Important Functions I

Absolute Value Function

Definition

The *absolute value* function, denoted $|x|$ is a function $f : \mathbb{R} \to \{y \in \mathbb{R} \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Floor & Ceiling Functions

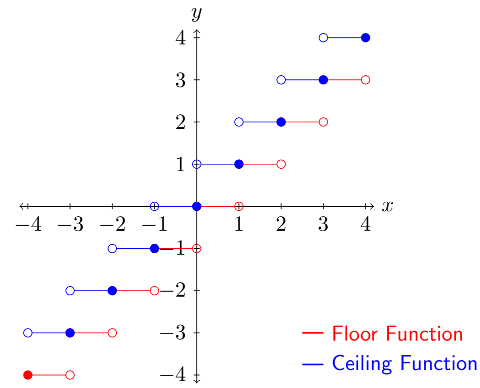
Definition

The *floor function*, denoted $\lfloor x \rfloor$ is a function $\mathbb{R} \to \mathbb{Z}$. Its value is the largest integer that is less than or equal to x.

The *ceiling function*, denoted $\lceil x \rceil$ is a function $\mathbb{R} \to \mathbb{Z}$. Its value is the smallest integer that is greater than or equal to x.

Floor & Ceiling Functions

Graphical View



Factorial Function

The factorial function gives us the number of permutations (that is, uniquely ordered arrangement) of a collection of n objects.

Definition

The *factorial function*, denoted $n!$ is a function $\mathbb{N} \to \mathbb{Z}^+$. Its value is the product of the first n positive integers.

$$n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

Factorial Function

Stirling's Approximation

The factorial function is defined on a discrete domain. In many applications, it is useful to consider a continuous version of the function (say if we want to differentiate it).

To this end, we have *Stirling's Formula*:

$$n! \approx \sqrt{2\pi n}\frac{n^n}{e^n}$$