Combinatorics

Computer Science & Engineering 235: Discrete Mathematics

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Combinatorics I

Introduction

Combinatorics is the study of collections of objects. Specifically, *counting* objects, arrangement, derangement, etc. of objects along with their mathematical properties.

Counting objects is important in order to analyze algorithms and compute discrete probabilities.

Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability.

In addition, combinatorics can be used as a proof technique.

A *combinatorial proof* is a proof method that uses counting arguments to prove a statement.

Combinatorics I

Motivating Example

How many arrangements are there of a deck of 52 cards?

The standard deck (The Mameluke deck) is thought to be 1000 years old. Have all possible $52!$ been dealt?

Suppose that 5 billion people have dealt 1 hand every second for the last 1000 years. Percentage of deals that have occurred:

$$\frac{5 \times 10^9 \cdot 1000 \cdot 365.25 \cdot 24 \cdot 60 \cdot 60}{52!} \approx 1.9562 \times 10^{-48}$$

To even deal 1% of all hands, we would require 5.11×10^{48} years (quindecillion).

Combinations

- Choosing k elements from a set of cardinality n is a *combination*
- Notations:

$$C_n^k = C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- Combinations are unordered
- Common usage: choosing singletons: $\binom{n}{1} = n$
- Choosing pairs: $\binom{n}{2} = \frac{n(n-1)}{2}$

Permutations

- Arranging n elements is a *permutation*
- Number of permutations:

$$n!$$

- Permutations are *ordered*

Product Rule

If two events are not mutually exclusive (that is, we do them separately), then we apply the product rule.

Theorem (Product Rule)

Suppose a procedure can be accomplished with two disjoint subtasks. If there are n_1 ways of doing the first task and n_2 ways of doing the second, then there are

$$n_1 \cdot n_2$$

ways of doing the overall procedure.

Sum Rule I

If two events *are* mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule.

Theorem (Sum Rule)

If an event e_1 can be done in n_1 ways and an event e_2 can be done in n_2 ways and e_1 and e_2 are mutually exclusive, then the number of ways of both events occurring is

$$n_1 + n_2$$

Sum Rule II

There is a natural generalization to any sequence of m tasks; namely the number of ways m mutually exclusive events can occur is

$$n_1 + n_2 + \cdots n_{m-1} + n_m$$

We can give another formulation in terms of sets. Let $A_1, A_2, \ldots, A_m$ be pairwise *disjoint* sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

In fact, this is a special case of the general *Principle of Inclusion-Exclusion*.

Principle of Inclusion-Exclusion (PIE) I

Introduction

Say there are two events, e_1 and e_2 for which there are n_1 and n_2 possible outcomes respectively.

Now, say that only *one* event can occur, not both.

In this situation, we cannot apply the sum rule? Why?

Principle of Inclusion-Exclusion (PIE) II

Introduction

We cannot use the sum rule because we would be *over counting* the number of possible outcomes.

Instead, we have to count the number of possible outcomes of e_1 and e_2 *minus* the number of possible outcomes in common to both; i.e. the number of ways to do both "tasks".

If again we think of them as sets, we have

$$|A_1| + |A_2| - |A_1 \cap A_2|$$

Principle of Inclusion-Exclusion (PIE) III

Introduction

More generally, we have the following.

Lemma

Let A, B be subsets of a finite set U. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cap B| \leq \min\{|A|, |B|\}$
3. $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$
4. $|\overline{A}| = |U| - |A|$
5. $|A \oplus B| = |A \cup B| - |A \cap B| = A + B - 2|A \cap B| = |A \setminus B| + |B \setminus A|$
6. $|A \times B| = |A| \cdot |B|$

Principle of Inclusion-Exclusion (PIE) I

Theorem

Theorem

Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_i |A_i| \\ &\quad - \sum_{i<j} |A_i \cap A_j| \\ &\quad + \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\ &\quad - \cdots \\ &\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \end{aligned}$$

Principle of Inclusion-Exclusion (PIE) II

Theorem

Each summation is over all i, pairs i, j with $i < j$, triples i, j, k with $i < j < k$ etc.

Principle of Inclusion-Exclusion (PIE) III

Theorem

To illustrate, when $n = 3$, we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad -\big[|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\big] \\ &\quad +|A_1 \cap A_2 \cap A_3| \end{aligned}$$

Principle of Inclusion-Exclusion (PIE) IV

Theorem

To illustrate, when $n = 4$, we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| \\ &\quad -\Big[|A_1 \cap A_2| + |A_1 \cap A_3| + +|A_1 \cap A_4| \\ &\qquad |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|\Big] \\ &\quad +\Big[|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \\ &\qquad |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|\Big] \\ &\quad -|A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

Principle of Inclusion-Exclusion (PIE) I

Example I

Example

How many integers between 1 and 300 (inclusive) are

1. Divisible by at least one of 3, 5, 7?
2. Divisible by 3 and by 5 but not by 7?
3. Divisible by 5 but by neither 3 nor 7?

Let

$$\begin{aligned} A &= \{n \mid 1 \leq n \leq 300 \wedge 3 \mid n\} \\ B &= \{n \mid 1 \leq n \leq 300 \wedge 5 \mid n\} \\ C &= \{n \mid 1 \leq n \leq 300 \wedge 7 \mid n\} \end{aligned}$$

Principle of Inclusion-Exclusion (PIE) II

Example I

How big are each of these sets? We can easily use the floor function;

$$\begin{aligned} |A| &= \lfloor 300/3 \rfloor = 100 \\ |B| &= \lfloor 300/5 \rfloor = 60 \\ |C| &= \lfloor 300/7 \rfloor = 42 \end{aligned}$$

For (1) above, we are asked to find $|A \cup B \cup C|$.

Principle of Inclusion-Exclusion (PIE) III

Example I

By the principle of inclusion-exclusion, we have that

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -\Big[|A \cap B| + |A \cap C| + |B \cap C|\Big] \\ &\quad +|A \cap B \cap C| \end{aligned}$$

It remains to find the final 4 cardinalities.

All three divisors, 3, 5, 7 are relatively prime. Thus, any integer that is divisible by *both* 3 and 5 must simply be divisible by 15.

Principle of Inclusion-Exclusion (PIE) IV

Example I

Using the same reasoning for all pairs (and the triple) we have

$$\begin{aligned}|A \cap B| &= \lfloor 300/15 \rfloor = 20\\ |A \cap C| &= \lfloor 300/21 \rfloor = 14\\ |B \cap C| &= \lfloor 300/35 \rfloor = 8\\ |A \cap B \cap C| &= \lfloor 300/105 \rfloor = 2\end{aligned}$$

Therefore,

$$|A \cup B \cup C| = 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162$$

Principle of Inclusion-Exclusion (PIE) V

Example I

For (2) above, it is enough to find

$$|(A \cap B) \setminus C|$$

By the definition of set-minus,

$$|(A \cap B) \setminus C| = |A \cap B| - |A \cap B \cap C| = 20 - 2 = 18$$

Principle of Inclusion-Exclusion (PIE) VI

Example I

For (3) above, we are asked to find

$$|B \setminus (A \cup C)| = |B| - |B \cap (A \cup C)|$$

By distributing B over the intersection, we get

$$\begin{aligned}|B \cap (A \cup C)| &= |(B \cap A) \cup (B \cap C)|\\ &= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)|\\ &= |B \cap A| + |B \cap C| - |B \cap A \cap C|\\ &= 20 + 8 - 2 = 26\end{aligned}$$

So the answer is $|B| - 26 = 60 - 26 = 34$.

Principle of Inclusion-Exclusion (PIE) I

Example II

The principle of inclusion-exclusion can be used to count the number of onto functions.

- Let $A = \{a_1, a_2, \ldots, a_m\}, |A| = m$
- Let $B = \{b_1, b_2, \ldots, b_n\}, |B| = n$
- Say $m \geq n$ (otherwise no onto functions exist)
- Observe: total number of functions is n^m
- Consider all functions such that no element maps to b_1:

$$(n-1)^m$$

- Generalize this: consider all functions such that no element maps to b_i for a particular i

Principle of Inclusion-Exclusion (PIE) II

Example II

- There are n such choices:

$$\binom{n}{1}$$

- Thus, the number of functions that do not map to (at least) a single element is:

$$\binom{n}{1}(n-1)^m$$

- We've over counted though: when we exclude b_2, then we are recounting functions that also exclude b_1
- Need to restore counts: consider pairs
- Consider all functions such that no element maps to a *pair* of elements:

$$\binom{n}{2}(n-2)^m$$

Principle of Inclusion-Exclusion (PIE) III

Example II

- But observe: the first equation is for functions that do not map to *at least* one element
- The second equation is for functions that do not map to *at least* two elements
- I.e. the first equation took away too many functions; the second restores this count
- Continuing for $i = 3, \ldots, n-1$ we can generalize this.

Theorem

Principle of Inclusion-Exclusion (PIE) IV
Example II

Let A, B be non-empty sets of cardinality m, n with $m \geq n$. Then there are

$$\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^m$$

i.e.

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m$$

onto functions $f : A \to B$.

Principle of Inclusion-Exclusion (PIE) V
Example II

This is related to *Stirling Numbers of the Second Kind*

Definition

Stirling numbers of the second kind represent the number of ways that you can partition n elements into k non-empty subsets and is defined as

$$S(n,k) = \left\{ {n \atop k} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$

- Only difference: for $j = k$, $(k-j)^n = 0$
- Stirling numbers *partition* into unordered subsets
- Function mapping requires ordering n mapped elements
- $n!$ possible mappings, so the $\frac{1}{k!}$ cancels out

Principle of Inclusion-Exclusion (PIE) VI
Example II

Example

How many ways of giving out 6 pieces of candy to 3 children if each child must receive at least one piece?

This can be modeled by letting A represent the set of candies and B be the set of children.

Then a function $f : A \to B$ can be interpreted as giving candy a_i to child c_j.

Since each child must receive at least one candy, we are considering only onto functions.

Principle of Inclusion-Exclusion (PIE) VII
Example II

To count how many there are, we apply the theorem and get (for $m = 6, n = 3$),

$$3^6 - \binom{3}{1}(3-1)^6 + \binom{3}{2}(3-2)^6 = 540$$

Derangements I

Consider the hatcheck problem.

- An employee checks hats from n customers.
- However, he forgets to tag them.
- When customer's check-out their hats, they are given one at random.

What is the probability that no one will get their hat back?

Derangements II

This can be modeled using *derangements*: permutations of objects such that no element is in its original position.

Theorem

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n\frac{1}{n!}\right]$$

Derangements III

Thus, the answer to the hatcheck problem is

$$\frac{D_n}{n!}$$

Its interesting to note that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \cdots$$

So that the probability of the hatcheck problem converges;

$$\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1} = .3679\ldots$$

Derangements IV

The formula for derangements can be derived as follows.

The number of derangements is equal to the number of permutation ($n!$) minus any permutation that leaves at least one element in its original place (non derangements).

- Total number of permutations: $n!$
- Number of permutations such that at least 1 element is in place:
$$-\binom{n}{1} \cdot (n-1)! = \frac{n!}{1!}$$
- We've over corrected; add back permutations such that at least 2 elements are in place:
$$+\binom{n}{2} \cdot (n-2)! = \frac{n!}{2!}$$

Derangements V

- In general:
$$(-1)^k \cdot \binom{n}{k} \cdot (n-k)! = \frac{n!}{k!}$$
- Last term will be when $k = n$ which is the identity permutation

The Pigeonhole Principle I

The *pigeonhole principle* states that if there are more pigeons than there are roosts (pigeonholes), for at least one pigeonhole, at least two pigeons must be in it.

Theorem (Pigeonhole Principle)

If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more objects.

This is a fundamental tool of elementary discrete mathematics. It is also known as the *Dirichlet Drawer Principle*.

The Pigeonhole Principle II

It is *seemingly* simple, but *very* powerful.

The difficulty comes in where and how to apply it.

Some simple applications in computer science:

- Calculating the probability of Hash functions having a collision.
- Proving that there can be *no* lossless compression algorithm compressing all files to within a certain ratio.

Lemma

For two finite sets A, B there exists a bijection $f : A \to B$ if and only if $|A| = |B|$.

Generalized Pigeonhole Principle I

Theorem

If N objects are placed into k boxes then there is at least one box containing at least

$$\left\lceil \frac{N}{k} \right\rceil$$

Example

In any group of 367 or more people, at least two of them must have been born on the same date.

Probablistic Pigeonhole Principle I

A probabilistic generalization states that if n objects are randomly put into m boxes with uniform probability (each object is placed in a given box with probability $1/m$) then at least one box will hold more than one object with probability,

$$1 - \frac{m!}{(m-n)!m^n}$$

Probablistic Pigeonhole Principle II

Example

Among 10 people, what is the probability that two or more will have the same birthday?

Here, $n = 10$ and $m = 365$ (ignore leapyears). Thus, the probability that two will have the same birthday is

$$1 - \frac{365!}{(365-10)!365^{10}} \approx .1169$$

So less than a 12% probability!

Only 23 people required for a better than 50% (50.7%) probability

With only 57, we have a better than 99% probability

Probablistic Pigeonhole Principle III

How many people do we need to have a better than 50% probability?

For what n is

$$1 - \frac{365!}{(365-n)!365^n} \geq .50$$

Surprisingly small: for $n = 23$, probability is greater than 50.7%!

Known as the "Birthday paradox"

Probablistic Pigeonhole Principle IV

Derivation: consider the n-permutations of m pigeonholes:

$$P(m, n) = \frac{m!}{(m-n)!} = m(m-1)(m-2)\cdots(m-n+1)$$

These are the pigeonholes that we will evenly distribute n objects into. Order is important because the objects are distinct.

We place each object into distinct pigeonhole with probability $\frac{1}{m}$, so in total:

$$\left(\frac{1}{m}\right)^n = \frac{1}{m^n}$$

Thus the probability is:

$$1 - \frac{m!}{(m-n)!m^n}$$

Probablistic Pigeonhole Principle V

Alternatively: consider choosing n bins (from a total of m bins to map n objects to:

$$\binom{m}{n} = \frac{n!}{(m-n)!n!}$$

But now consider actually mapping objects $o_1, \ldots, o_n$ to each bin.

Here, order matters, but in addition, we only want to map one object to one bucket. That is, we want to count the total number of one-to-one functions from $o_1, \ldots, o_n$ to the n bins we chose. Thus:

$$\frac{n!}{(m-n)!n!} \cdot n! = \frac{n!}{(m-n)!}$$

Again, viewing allocation of objects to buckets, how many functions in total are there?

$$m^n$$

Probablistic Pigeonhole Principle VI

Thus the probability of a random allocation resulting in all bins having 0 or 1 objects is

$$\frac{n!}{(m-n)!} \div m^n = \frac{m!}{(m-n)!m^n}$$

Pigeonhole Principle I

Example I

Example

Show that in a room of n people with certain acquaintances, some pair must have the same number of acquaintances.

Note that this is equivalent to showing that any symmetric, irreflexive relation on n elements must have two elements with the same number of relations.

We'll show by contradiction using the pigeonhole principle.

Assume to the contrary that every person has a different number of acquaintances; $0, 1, \ldots, n-1$ (we cannot have n here because it is irreflexive). Are we done?

Pigeonhole Principle II

Example I

No, since we only have n people, this is okay (i.e. there are n possibilities).

We need to use the fact that acquaintanceship is a symmetric, irreflexive relation.

In particular, some person knows 0 people while another knows $n-1$ people.

In other words, someone knows everyone, but there is also a person that knows no one.

Thus, we have reached a contradiction. □

Pigeonhole Principle I

Example II

Example

Show that in any list of ten nonnegative integers, $a_0, \ldots, a_9$, there is a string of consecutive items of the list $a_l, a_{l+1}, \ldots, a_k$ whose sum is divisible by 10.

Consider the following 10 numbers.

$$\begin{array}{l} a_0 \\ a_0 + a_1 \\ a_0 + a_1 + a_2 \\ \vdots \\ a_0 + a_1 + a_2 + \cdots + a_9 \end{array}$$

If any one of them is divisible by 10 then we are done.

Pigeonhole Principle II

Example II

Otherwise, we observe that each of these numbers must be in one of the congruence classes

$$1 \textbf{ mod } 10, 2 \textbf{ mod } 10, \ldots, 9 \textbf{ mod } 10$$

By the pigeonhole principle, at least two of the integers above must lie in the same congruence class. Say a, a' lie in the congruence class $k \textbf{ mod } 10$.

Then

$$(a - a') \equiv k - k (\text{mod } 10)$$

and so the difference $(a - a')$ is divisible by 10. □

Pigeonhole Principle I

Example III

Example

Say 30 buses are to transport 2000 Cornhusker fans to Colorado. Each bus has 80 seats. Show that

1. One of the buses will have 14 empty seats.
2. One of the buses will carry at least 67 passengers.

For (1), the total number of seats is $30 \cdot 80 = 2400$ seats. Thus there will be $2400 - 2000 = 400$ empty seats total.

Pigeonhole Principle II

Example III

By the generalized pigeonhole principle, with 400 empty seats among 30 buses, one bus will have at least

$$\left\lceil \frac{400}{30} \right\rceil = 14$$

empty seats.

For (2) above, by the pigeonhole principle, seating 2000 passengers among 30 buses, one will have at least

$$\left\lceil \frac{2000}{30} \right\rceil = 67$$

passengers.

Permutations I

A *permutation* of a set of distinct objects is an *ordered* arrangement of these objects. An ordered arrangement of r elements of a set is called an r-*permutation*.

Theorem

The number of r *permutations of a set with* n *distinct elements is*

$$P(n, r) = \prod_{i=0}^{r-1}(n - i) = n(n-1)(n-2)\cdots(n-r+1)$$

Permutations II

It follows that

$$P(n, r) = \frac{n!}{(n-r)!}$$

In particular,

$$P(n, n) = n!$$

Again, note here that *order is important*. It is necessary to distinguish in what cases order is important and in which it is not.

Permutations
Example I

Example

How many pairs of dance partners can be selected from a group of 12 women and 20 men?

The first woman can be partnered with any of the 20 men. The second with any of the remaining 19, etc.

To partner all 12 women, we have

$$P(20, 12)$$

Another perspective: choose 12 men to include, *then* order them

Permutations I
Variation

What if we allowed all 32 people to pair up in any combination?

- Number of permutations: $32!$
- But a given pair, AB is the same as BA
- Correct for each pair: 2^{16}
- Now each pair's ordering also doesn't matter
- Correct for each such permutation: $16!$

$$\frac{32!}{2^{16}16!}$$

Permutations II
Variation

Generalization: given kn objects, how many ways are there to form n groups of size k?

$$\frac{(nk)!}{(k!)^n \cdot n!}$$

Permutations
Example II

Example

In how many ways can the English letters be arranged so that there are exactly ten letters between a and z?

The number of ways of arranging 10 letters between a and z is $P(24, 10)$. Since we can choose either a or z to come first, there are $2P(24, 10)$ arrangements of this 12-letter block.

For the remaining 14 letters, there are $P(15, 15) = 15!$ arrangements. In all, there are

$$2P(24, 10) \cdot 15!$$

Permutations

Example III

Example

How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern bge nor eaf?

The number of total permutations is $P(7, 7) = 7!$.

If we fix the pattern bge, then we can consider it as a single block. Thus, the number of permutations with this pattern is $P(5, 5) = 5!$.

Permutations

Example III - Continued

Fixing the pattern eaf we have the same number, $5!$.

Thus we have

$$7! - 2(5!)$$

Is this correct?

No. We have taken away too many permutations: ones containing *both* eaf and bge.

Here there are two cases, when eaf comes first and when bge comes first.

Permutations

Example III - Continued

eaf cannot come before bge, so this is not a problem.

If bge comes first, it must be the case that we have $bgeaf$ as a single block and so we have 3 blocks or $3!$ arrangements.

Altogether we have

$$7! - 2(5!) + 3! = 4806$$

Combinations I

Definition

Whereas permutations consider order, *combinations* are used when *order does not matter*.

Definition

A k-combination of elements of a set is an unordered selection of k elements from the set. A combination is simply a subset of cardinality k.

Combinations II

Definition

Theorem

The number of k-combinations of a set with cardinality n with $0 \leq k \leq n$ is

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Note: the notation, $\binom{n}{k}$ is read, "n choose k". In TEX use `{n choose k}` (with the forward slash).

Combinations III

Definition

A useful fact about combinations is that they are symmetric.

$$\binom{n}{1} = \binom{n}{n-1}$$

$$\binom{n}{2} = \binom{n}{n-2}$$

etc.

Combinations IV
Definition

This is formalized in the following corollary.

Corollary

Let n, k be nonnegative integers with $k \leq n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinations I
Example I

Example

In the Powerball lottery, you pick five numbers between 1 and 59 and a single "powerball" number between 1 and 35. How many possible plays are there?

Order here doesn't matter, so the number of ways of choosing five regular numbers is

$$\binom{59}{5}$$

Combinations II
Example I

We can choose among 35 power ball numbers. These events are not mutually exclusive, thus we use the product rule.

$$35 \cdot \binom{59}{5} = 35\frac{59!}{(59-5)!5!} = 175,223,510$$

So the odds of winning are

$$\frac{1}{175,223,510} < 5.70699 \times 10^{-9} = .000000005706996\,75\%$$

Combinations I
Example II

Example

In a sequence of 10 coin tosses, how many ways can 3 heads and 7 tails come up?

The number of ways of choosing 3 heads out of 10 coin tosses is

$$\binom{10}{3}$$

Combinations II
Example II

However, this is the same as choosing 7 tails out of 10 coin tosses;

$$\binom{10}{3} = \binom{10}{7} = 120$$

This is a perfect illustration of the previous corollary.

Combinations III
Example II

Another perspective: what is the corresponding probability of 10 coin tosses ending up with 3 heads/7 tails?

It is the number of such outcomes divided by the total number of possible outcomes:

$$\frac{C(10,7)}{2^{10}}$$

Why 2^{10}? Each toss was an independent event with 2 possible outcomes.

Another perspective: the total number of outcomes is equal to the total number of ways to choose: (0 tails, 10 heads), (1 tails, 9 heads), (2 tails, 8 heads), ... (10 tails, 0 heads)

Which is:

$$\sum_{i=0}^{10} \binom{10}{i} = 2^{10}$$

Combinations IV

Example II

That is, the sum of binomial coefficients is equal to 2^n

Gambler's Fallacy

Say that we've flipped a coin and heads has appeared 9 times in a row.

What is the probability that the next flip will be tails? Heads?

Each event is *independent*: there is a fundamental difference between probabilities involving a sequence of events (parlays) and the probability of an event *given* a prior sequence.

August 18, 1913 Monte Carlo casino: black appeared 15 times in a row. Gamblers rushed to bet on red. Black would appear a total of 26 times in a row; many lost their bets thinking that red was "due".

Krusty on why he bet against the Harlem Globetrotters: "I thought the Generals were due!"

Combinations I

Example III

Example

How many possible committees of five people can be chosen from 20 men and 12 women if

1. if exactly three men must be on each committee?
2. if at least four women must be on each committee?

Combinations II

Example III

For (1), we must choose 3 men from 20 then two women from 12. These are not mutually exclusive, thus the product rule applies.

$$\binom{20}{3}\binom{12}{2}$$

Combinations III

Example III

For (2), we consider two cases; the case where four women are chosen and the case where five women are chosen. These two cases *are* mutually exclusive so we use the addition rule.

For the first case we have

$$\binom{20}{1}\binom{12}{4}$$

Combinations IV

Example III

And for the second we have

$$\binom{20}{0}\binom{12}{5}$$

Together we have

$$\binom{20}{1}\binom{12}{4} + \binom{20}{0}\binom{12}{5} = 10,692$$

Binomial Coefficients I

Introduction

The number of r-combinations, $\binom{n}{r}$ is also called a *binomial coefficient*.

They are the coefficients in the expansion of the expression (multivariate polynomial), $(x+y)^n$. A *binomial* is a sum of two terms.

Binomial Coefficients II

Introduction

Theorem (Binomial Theorem)

Let x, y be variables and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Binomial Coefficients III

Introduction

Expanding the summation, we have

$$\begin{aligned}(x+y)^n &= \tbinom{n}{0}x^n + \tbinom{n}{1}x^{n-1}y + \tbinom{n}{2}x^{n-2}y^2 + \cdots \\ &\quad + \tbinom{n}{n-1}xy^{n-1} + \tbinom{n}{n}y^n\end{aligned}$$

For example,

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= (x+y)(x^2+2xy+y^2) \\ &= x^3+3x^2y+3xy^2+y^3)\end{aligned}$$

Binomial Coefficients I

Example

Example

What is the coefficient of the term x^8y^{12} in the expansion of $(3x+4y)^{20}$?

By the Binomial Theorem, we have

$$(3x+4y)^n = \sum_{j=0}^{20} \binom{20}{j} (3x)^{20-j}(4y)^j$$

So when $j = 12$, we have

$$\binom{20}{12}(3x)^8(4y)^{12}$$

so the coefficient is $\frac{20!}{12!8!}3^8 4^{12} = 13866187326750720$.

Binomial Coefficients I

More

A lot of useful identities and facts come from the Binomial Theorem.

Corollary

Binomial Coefficients II

More

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} &= 2^n \\ \sum_{k=0}^{n}(-1)^k\binom{n}{k} &= 0 \quad n \geq 1 \\ \sum_{k=0}^{n}x^k\binom{n}{k} &= (1+x)^n \\ \sum_{k=0}^{n}2^k\binom{n}{k} &= 3^n\end{aligned}$$

And many more.

Binomial Coefficients III

More

Most of these can be proven by either induction or by a combinatorial argument.

Theorem (Vandermonde's Identity)

Let m, n, r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Binomial Coefficients IV

More

Corollary

If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Corollary

Let n, r be nonnegative integers, $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Binomial Coefficients I

Pascal's Identity & Triangle

The following is known as Pascal's Identity which gives a useful identity for efficiently computing binomial coefficients.

Theorem (Pascal's Identity)

Let $n, k \in \mathbb{Z}^+$ with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's Identity forms the basis of a geometric object known as Pascal's Triangle.

Pascal's Triangle

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\iddots \qquad\qquad\qquad\qquad \ddots
\end{array}$$

Pascal's Triangle

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
\iddots \qquad\qquad\qquad\qquad \ddots
\end{array}$$

Pascal's Triangle

$$\binom{3}{2} + \binom{3}{3} = \binom{4}{3}$$

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\iddots \qquad\qquad\qquad\qquad \ddots
\end{array}$$

Generalized Permutations I

Sometimes we are concerned with permutations and combinations in which *repetitions* are allowed.

Theorem

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Generalized Combinations I

Theorem

There are

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

r-combinations from a set with n elements when repetition of elements is allowed.

Generalized Combinations II

To see this consider:

- $n-1$ bars (so n "cells" that represent that types of items)
- r stars: (if x stars are placed into a given cell, its the number of elements of that type that we choose)
- So $n-1+r$ "things" to be arranged
- However, we've over counted: stars and bars are indistinguishable
- Any sequence of contiguous stars/bars is the same under any ordering, so we need to divide out by permutations of $n-1$ *and* r:

$$\frac{(n-1+r)!}{(n-1)!r!} = \frac{(n-1+r)!}{(n+r-1-r)!r!} = \binom{n+r-1}{r}$$

Generalized Combinations I

Example

Example

There are 30 varieties of donuts from which we wish to buy a dozen. How many possible orders are there?

Here $n = 30$ and we wish to choose $r = 12$. Order does not matter and repetitions are possible, so we apply the previous theorem to get that there are

$$\binom{30+12-1}{12}$$

possible orders.

Generalized Combinations II

Example

Theorem

The number of different permutations of n objects where there are n_1 indistinguishable objects of type 1, n_2 of type 2, ..., and n_k of type k is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

An equivalent way of interpreting this theorem is the number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i for $i = 1, 2, \ldots, k$.

Generalized Combinations III

Example

Example

How many permutations of the word "Mississippi" are there?

"Mississippi" contains 4 distinct letters, M, i, s and p; with $1, 4, 4, 2$ occurrences respectively.

Therefore there are

$$\frac{11!}{1!4!4!2!}$$

permutations.

Distinguishable Objects into Distinguishable Boxes

Example: how many ways are there to deal a 52 card deck into 5 card hands to four players?

$$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}$$

Theorem

The number of ways to distribute n distinguishable objects into k distinguishable boxes such that each box has n_i objects for $i = 1, \ldots, k$ is

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

Card example: the 5 box should contain the remaining 32 undealt cards

Indistinguishable Objects into Distinguishable Boxes

Indistinguishable objects into distinguishable boxes: equivalent to n-combinations of k elements when repetition is allowed (parameters are switched).

Theorem

The number of ways to distribute n indistinguishable objects into k distinguishable boxes is

$$\binom{k+n-1}{n} = \binom{k+n-1}{k-1}$$

Distinguishable Objects into Indistinguishable Boxes

Distinguishable objects into indistinguishable boxes: not the same as vice versa.

Equivalent to Stirling numbers of the 2nd kind (the $\frac{1}{k!}$ factor recognizes that boxes are indistinguishable and corrects for it)

Theorem

The number of ways to distribute n distinguishable objects into k indistinguishable boxes is

$$\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$

Indistinguishable Objects into Indistinguishable Boxes I

- Far more complicated
- Example: 6 copies of the same book into 4 identical boxes: $(6), (5, 1), (4, 2), (4, 1, 1), (3, 3), \ldots$
- Equivalent to *partitions* of an integer sum
- No closed form, requires a generating function:

Generating Function Example I

How many ways are there to make change for a dollar (using half-dollars, quarters, dimes, nickels, pennies)?

The generating function:

$$C(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}$$

gives us the answer: for any amount of change c, we want the coefficient of x^c in the expansion of $C(x)$.

For $c = 100$, the coefficient of x^{100} is 292.

Related: Set Partitions I

The number of ways to partition a *set* into disjoint non-empty subsets (such that their union is the original set).

Example: $S = \{1, 2, 3\}$ then the partitions are:

$$\begin{gathered}\{\{a\},\{b\},\{c\}\}\\ \{\{a\},\{b,c\}\}\\ \{\{b\},\{a,c\}\}\\ \{\{c\},\{a,b\}\}\\ \{\{a,b,c\}\}\end{gathered}$$

Theorem

Related: Set Partitions II

The number of ways to partition a set of size n into disjoint non-empty subsets corresponds to the n-th Bell Number:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Note: $B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, 52, 203, 877, 4140, 21147, 115975, \ldots$

Basic Idea:

- Consider a new element e_{n+1}
- It could be in its own partition, thus there are $\binom{n}{0} \cdot B_n$ partitionings of the remaining elements

Related: Set Partitions III

- It could be paired with one other element, leaving $\binom{n}{1} \cdot B_{n-1}$ partitionings of remaining elements
- It could be paired with two other element, leaving $\binom{n}{2} \cdot B_{n-2}$ partitionings of remaining elements
- All the way to $\binom{n}{n} \cdot B_0$

Each Bell number is the sum of Stirling numbers of the second kind:

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

Idea: Stirling numbers give the number of ways to partition into k non-empty subsets; summing over all such k gives us the formula.

Generating Permutations & Combinations I

Introduction

In general, it is inefficient to solve a problem by considering all permutations or combinations since there are an exponential number of such arrangements.

Nevertheless, for many problems, *no better approach is known*. When exact solutions are needed, *back-tracking* algorithms are used.

Generating permutations or combinations are sometimes the basis of these algorithms.

Generating Permutations & Combinations II

Introduction

Example (Traveling Sales Person Problem)

Consider a salesman that must visit n different cities. He wishes to visit them in an order such that his overall distance traveled is minimized.

Generating Permutations & Combinations III

Introduction

This problem is one of hundreds of NP-complete problems for which no known efficient algorithms exist. Indeed, it is believed that *no* efficient algorithms exist. (Actually, Euclidean TSP is not even known to be in NP!)

The only known way of solving this problem *exactly* is to try all $n!$ possible routes.

We give several algorithms for generating these combinatorial objects.

Generating Combinations I

Recall that combinations are simply all possible subsets of size r. For our purposes, we will consider generating subsets of

$$\{1, 2, 3, \ldots, n\}$$

The algorithm works as follows.

- Start with $\{1, \ldots, r\}$
- Assume that we have $a_1 a_2 \cdots a_r$, we want the next combination.
- Locate the last element a_i such that $a_i \neq n - r + i$.
- Replace a_i with $a_i + 1$.
- Replace a_j with $a_i + j - i$ for $j = i + 1, i + 2, \ldots, r$.

Generating Combinations II

The following is pseudocode for this procedure.

Algorithm (Next r-Combination)

```
INPUT   : A set of n elements and an r-combination, a_1 ··· a_r.
OUTPUT  : The next r-combination.
i = r
WHILE a_i = n − r + i DO
    i = i − 1
END
a_i = a_i + 1
FOR j = (i + 1) ... r DO
    a_j = a_i + j − i
END
```

Generating Combinations III

Example

Find the next 3-combination of the set $\{1, 2, 3, 4, 5\}$ after $\{1, 4, 5\}$

Here, $a_1 = 1, a_2 = 4, a_3 = 5$.

The last i such that $a_i \neq 5 - 3 + i$ is 1.

Thus, we set

$$\begin{aligned} a_1 &= a_1 + 1 = 2 \\ a_2 &= a_1 + 2 - 1 = 3 \\ a_3 &= a_1 + 3 - 1 = 4 \end{aligned}$$

So the next r-combination is $\{2, 3, 4\}$.

Generating Permutations

The text gives an algorithm to generate permutations in lexicographic order. Essentially the algorithm works as follows.

Given a permutation,

- Choose the left-most pair a_j, a_{j+1} where $a_j < a_{j+1}$.
- Choose the least item to the right of a_j greater than a_j.
- Swap this item and a_j.
- Arrange the remaining (to the right) items in order.

Generating Permutations

Lexicographic Order

Algorithm (Next Permutation (Lexicographic Order))

```
INPUT   : A set of n elements and an r-permutation, a_1 ··· a_r.
OUTPUT  : The next r-permutation.
j = n − 1
WHILE a_j > a_{j+1} DO
    j = j − 1
END
//j is the largest subscript with a_j < a_{j+1}
k = n
WHILE a_j > a_k DO
    k = k − 1
END
//a_k is the smallest integer greater than a_j to the right of a_j
swap(a_j, a_k)
r = n
s = j + 1
WHILE r > s DO
    swap(a_r, a_s)
```

Generating Permutations I

Often there is no reason to generate permutations in lexicographic order. Moreover, even though generating permutations is inefficient in itself, lexicographic order induces even *more* work.

An alternate method is to *fix* an element, then recursively permute the $n - 1$ remaining elements.

Another method has the following attractive properties.

- It is bottom-up (non-recursive).
- It induces a *minimal-change* between each permutation.

Generating Permutations II

The algorithm is known as the *Johnson-Trotter algorithm*.

We associate a direction to each element, for example:

$$\overrightarrow{3}\,\overleftarrow{2}\,\overrightarrow{4}\,\overleftarrow{1}$$

A component is *mobile* if its direction points to an adjacent component that is *smaller* than itself. Here 3 and 4 are mobile and 1 and 2 are not.

Generating Permutations III

Algorithm (JohnsonTrotter)

```
INPUT    : An integer n.
OUTPUT   : All possible permutations of ⟨1, 2, ... n⟩.
π ← ←1 ←2 ... ←n
WHILE There exists a mobile integer k ∈ π DO
    k ← largest mobile integer
    swap k and the adjacent integer k points to
    reverse direction of all integers > k
    Output π
END
```

Johnson-Trotter Examples

Example A: consider permutations of $(1, \ldots, 6)$:

$$\overleftarrow{4}, \overleftarrow{3}, \overleftarrow{1}, \overleftarrow{2}, \overrightarrow{6}, \overrightarrow{5}$$

2, 6 are mobile, so swap 6, 5; no orientation changes:

$$\overleftarrow{4}, \overleftarrow{3}, \overleftarrow{1}, \overleftarrow{2}, \overrightarrow{5}, \overrightarrow{6}$$

2 is the only mobile element, flip 1, 2; orientation of 4, 3, 5, 6 gets reversed:

$$\overrightarrow{4}, \overrightarrow{3}, \overleftarrow{2}, \overleftarrow{1}, \overleftarrow{5}, \overleftarrow{6}$$

More Examples

As always, the best way to learn new concepts is through practice and examples.

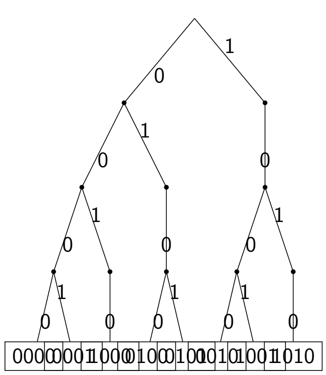
Example I I

Example

How many bit strings of length 4 are there such that 11 never appears as a substring?

We can represent the set of string graphically using a tree.

Example I II



Therefore, the number of such bit string is 8.

Example: Counting Functions I I

Example

Let S, T be sets such that $|S| = n, |T| = m$. How many functions are there mapping $f : S \to T$? How many of these functions are one-to-one?

A function simply maps each s_i to some t_j, thus for each n we can choose to send it to *any* of the elements in T.

Example: Counting Functions I II

Each of these is an independent event, so we apply the multiplication rule;

$$\underbrace{m \times m \times \cdots \times m}_{n \text{ times}} = m^n$$

If we wish f to be one-to-one, we must have that $n \leq m$, otherwise we can easily answer 0.

Now, each s_i must be mapped to a *unique* element in T. For s_1, we have m choices. However, once we have made a mapping (say t_j), we cannot map subsequent elements to t_j again.

Example: Counting Functions I III

In particular, for the second element, s_2, we now have $m - 1$ choices. Proceeding in this manner, s_3 will have $m - 2$ choices, etc. Thus we have

$$m \cdot (m-1) \cdot (m-2) \cdot \cdots \cdot (m-(n-2)) \cdot (m-(n-1))$$

An alternative way of thinking about this problem is by using the choose operator: we need to choose n elements from a set of size m for our mapping;

$$\binom{m}{n} = \frac{m!}{(m-n)!n!}$$

Example: Counting Functions I IV

Once we have chosen this set, we now consider all permutations of the mapping, i.e. $n!$ different mappings for this set. Thus, the number of such mappings is

$$\frac{m!}{(m-n)!n!} \cdot n! = \frac{m!}{(m-n)!}$$

Example: Counting Functions II

Recall this question from the 1st exam:

Example

Let $S = \{1, 2, 3\}, T = \{a, b\}$. How many onto functions are there mapping $S \to T$? How many one-to-one functions are there mapping $T \to S$?

Example: Counting Primes I

Example

Give an estimate for how many 70 bit primes there are.

Recall that the number of primes not more than n is about

$$\frac{n}{\ln n}$$

Using this fact, the number of primes not exceeding 2^{70} is

$$\frac{2^{70}}{\ln 2^{70}}$$

Example: Counting Primes II

However, we have over counted—we've counted 69-bit, 68-bit, etc primes as well.

The number of primes not exceeding 2^{69} is about

$$\frac{2^{69}}{\ln 2^{69}}$$

Thus the difference is

$$\frac{2^{70}}{\ln 2^{70}} - \frac{2^{69}}{\ln 2^{69}} \approx 1.19896 \times 10^{19}$$

Example: More sets I

Example

How many integers in the range $1 \leq k \leq 100$ are divisible by 2 or 3?

Let

$$
\begin{aligned}
A &= \{x \mid 1 \leq x \leq 100, 2 \mid x\} \\
B &= \{y \mid 1 \leq x \leq 100, 3 \mid y\}
\end{aligned}
$$

Clearly, $|A| = 50, |B| = \lfloor \frac{100}{3} \rfloor = 33$, so is it true that $|A \cup B| = 50 + 33 = 83$?

Example: More sets II

No; we've over counted again—any integer divisible by 6 will be in both sets. How much did we over count?

The number of integers between 1 and 100 divisible by 6 is $\lfloor \frac{100}{6} \rfloor = 16$, so the answer to the original question is

$$|A \cup B| = (50 + 33) - 16 = 67$$