Diffusion based Projection Method for Distributed Source Localization in Wireless Sensor Networks

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***Abstract*—Source localization based on energy measurements is an important problem in wireless sensor networks (WSNs). It is well known that the associated objective function is not convex and may have multiple local optima and saddle points. Most of the existing algorithms can not achieve global optima. In this paper, we formulate the source localization as a convex feasibility problem (CFP) and propose a diffusion based projection method as a solution. In the proposed method, no fusion center is required; the sensor nodes need to communicate only with their closest neighbors and all the sensors update their estimations simultaneously and finally they are able to achieve consensus on a possible minimizer asymptotically. The proposed method has low complexity and achieves global optimality. Theoretical analysis and simulation results show that the proposed method has good estimation performance whether or not the CFP is consistent or inconsistent.**

I. Introduction

Source localization in wireless sensor networks is an important problem encountered in acoustic networks [1], [2]. Energy based approach for acoustic source localization is an appropriate choice since the acoustic energy emitted by the sources usually varies slowly. As such the acoustic energy time series can be sampled at a much lower rate compared to the raw acoustic time series [1], [2]. Therefore, limited data needs to be transmitted among sensors via the often congested wireless communication channels. This reduces the energy consumption for data transmissions on individual sensor nodes and saves communication bandwidth over shared wireless channels.

Recently, many approaches with energy-based source localization have been proposed. A maximum likelihood method is proposed in [1] where a Multi-Resolution (MR) search is needed to find the optimal solution. In [2], an efficient expectation-maximization (EM) algorithm is proposed for multi-source localization problems. It can estimate the source locations individually and can efficiently avoid the local optima through effective sequential dominant-source (SDS) initialization and parameterized search methods. The main drawbacks of the methods of [1] and [2] are that they require the transmission of measurements from each node in the network to a central point for processing and have high computational complexity. In [3], Rabbat and Nowak proposed a distributed implementation of the incremental gradient (IG) algorithm to solve the nonlinear least-square problem. However the algorithm may fall into local optima. In [4], a two-stage algebraic closed-form solution is presented. The first stage computes the source location together with an auxiliary variable using weighted least squares method. The second stage explores the relationship between the source location and the auxiliary variable to improve the location estimation. The drawbacks of the methods in [3], [4], or any other least-square based methods, is that they are sensitive to local optima and saddle points and have a low estimation accuracy when the signal to noise ratio (SNR) is low.

In the literature, there is limited work on fully distributed approaches in which no fusion center is required, where sensor nodes need to communicate only with their closest neighbors and all the sensors update their estimations simultaneously. In this paper, we formulate the distributed source localization problem as a convex feasibility problem (CFP). The mathematical formulation of CFP is as follows.

Suppose in a Hilbert space, $C_1, \ldots, C_N$ are closed convex subsets with intersection C: $C = C_1 \cap \ldots \cap C_N$. *Convex feasibility problem (CFP):* Find some point x in C. We call the CFP consistent if $C \neq \emptyset$, and otherwise call it inconsistent. The projection based method is well studied and can be used for solving the CFP. Most existing projection methods are centralized and implemented in a sequential or parallel manner. In [5], Blatt and Hero propose to use the sequential projection method for source localization without the theoretical proof. In this paper, we also examine how to apply the parallel projection method to source localization.

However, there are few works on diffusion based projection method for CFP in the literature. In [9], the authors proposed a constrained consensus method for convex optimization problems, however it is restricted to the consistent cases and can not be used for source localization which is usually inconsistent. In this paper, we propose a general diffusion based projection protocol which can be applied to both the consistent and inconsistent CFP problems. We prove that with an appropriately selected step-size sequence, the estimate of all the sensors generated by the proposed algorithm will converge to the same global optimal solution. If the problem is consistent, the converged source location estimation will also lie in the intersection of convex sets determined by each sensor. For the inconsistent case, our diffusion method will converge to a point close to the true source location.

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II. Problem Formulation

The problem of interest is to determine the location of an active source in a sensor network. Assume the sensor field is denoted is by $S \in \mathbb{R}^2$. Let the source be located at an unknown coordinate pair $\theta = [x, y]^T$ and transmit at power level P. We assume that there are N sensor nodes performing sensing using energy detection. At the i-th sensor with its known coordinate $r_i = [x_i, y_i]^T, i = 1, 2, \ldots, N$, the received power can be written as

$$P_{r_i} = g_i \frac{P}{d_{i_s}^{\alpha}} + w_i, \tag{1}$$

where $d_{i_s} = \|\theta - r_i\|$ is the Euclidean distance between the i-th sensor and the source, g_i is the gain factor of i-th sensor. α is the power-loss factor (in this paper we assume $\alpha = 2$) and w_i is the receiver noise at the i-th sensor. We assume an AWGN channel with $w_i \sim N(0, \sigma_i^2), i = 1, 2, \ldots, N$. In this paper, we assume that the acoustic source power lever P is known. The only parameter we need to estimate is the source's location vector $\theta = [x, y]^T$.

The maximum likelihood estimator (MLE) is found by solving the nonlinear least square problem when the noise is Gaussian

$$\theta_{ML}^* = \arg\min \sum_{i=1}^{N} \left[P_{r_i} - g_i \frac{P}{d_{i_s}^2} \right]^2 = \arg\min \sum_{i=1}^{N} f_i(\theta). \tag{2}$$

Clearly, $f_i(\theta)$ attain its minimum 0 on the circle $C_i = \{\theta \in \mathbb{R}^2 : \|\theta - r_i\| = \sqrt{g_i P / P_{r_i}}\}$. However, because of the observation noise, the source may not appear on the circles$\mathcal{D}_i$ as the disk: $\mathcal{D}_i = \{\theta \in \mathbb{R}^2 : \|\theta - r_i\| \le \sqrt{g_i P / P_{r_i}}\}$.

Clearly, $\mathcal{D}_i$ forms a convex set which is a disk. It is easy to see that source localization problem can be solved by letting the estimator be a point in the intersection of the sets $\mathcal{D}_i, i = 1, 2, \ldots, N$. That is,

$$\hat{\theta} \in \mathcal{D} = \bigcap_{i=1}^{N} \mathcal{D}_i \subset \mathbb{R}^2. \tag{3}$$

Until now, we have formulated the source localization problems as a convex feasibility problem (CFP). However due to the observation noise, the feasibility problems may turn out to be inconsistent, i.e., the intersection $\mathcal{D}$ might be empty. An illustration for consistent and inconsistent cases is presented in Fig. 1.

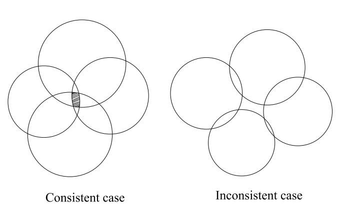


Fig. 1. Consistent case and inconsistent case

Since the convex feasibility problem may turn out to be inconsistent, finding a solution to this problem is equivalent to finding a point θ^* which minimizes the sum of the squares of the distances to the convex set $\mathcal{D}_i$s.

$$\theta^* = \arg\min_{\theta \in \mathbb{R}^2} \sum_{i=1}^{N} \|\theta - \mathcal{P}_{\mathcal{D}_i}(\theta)\|^2. \tag{4}$$

where for a close convex set $S \subseteq \mathbb{R}^2$ and vector $x \in \mathbb{R}^2, \mathcal{P}_S(x)$ is the orthogonal projection of x onto S. That is,

$$\mathcal{P}_S(x) = \arg\min_{y \in \mathbb{R}^2} \|x - y\|, y \in S. \tag{5}$$

For the source localization problem, the projection operator has a closed-form expression for (5) given as:

$$\mathcal{P}_{\mathcal{D}_i}(x) = \begin{cases} x, \|x - r_i\| \le \sqrt{g_i P / P_{r_i}}, \\ r_i + \sqrt{g_i P / P_{r_i}} \frac{x - r_i}{\|x - r_i\|}, \text{otherwise.} \end{cases} \tag{6}$$

We assume that G is the set of least-square solutions of the convex feasibility problem in (4).

Proposition 1: The set G is nonempty, closed, convex and bounded.

It can be easily checked that if the problem is consistent, i.e., $\mathcal{D} = \bigcap_{i=1}^{N} D_i \neq \emptyset$, then $\sum_{i=1}^{N} \|\hat{\theta} - \mathcal{P}_{\mathcal{D}_i}(\hat{\theta})\| = 0$ where $\hat{\theta} \in G$ and $G = \bigcap_{i=1}^{N} D_i$.

III. Sequential and Parallel Projection Methods

In this section, we first review the sequential projection method for source localization proposed by Blatt and Hero [5]. Then we propose to use the parallel projection method in the image processing area proposed in [8] which is a centralized algorithm to solve the problem. An understanding of these methods will be useful later in Section IV when we introduce our proposed distributed protocol.

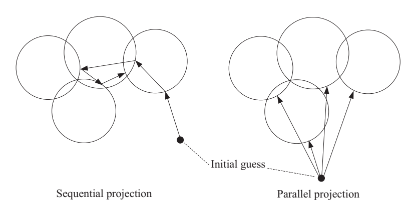


Fig. 2. Parallel and sequential projection methods

A. Sequential Projection Method

The sequential projection method, also termed POCS (projection on convex sets), is a cyclic algorithm. From Fig. 2, we can see that the data is processed across the sensor sequence. The update rule of the sequential projection method is given as follows

$$\theta(k+1) = \theta(k) + \lambda(k) \left[\mathcal{P}_{\mathcal{D}_{\tau(k)}}(\theta(k)) - \theta(k) \right] \tag{7}$$

where $\{\lambda(k)\}$ is a sequence of relaxation parameters satisfying for all $k, \epsilon_1 \le \lambda(k) \le 2 - \epsilon_2$ for some $\epsilon_1, \epsilon_2 > 0, \tau(k) = k \bmod N$.

Theorem 1: [7] If $\mathcal{D} = \bigcap_{i=1}^{N} D_i \neq \emptyset$, any sequence $\theta(k), k \ge 0$ converges to a point in $\mathcal{D}$. □

From Theorem 1, we can see that the POCS algorithm has good convergence performance in the consistent case of CFP.

However the convergence behavior of POCS in the inconsistent case is generally unsatisfactory. We need to design a new relaxation sequence $\lambda(k)_{(k\geq 0)}$, that is $\sum_{k=0}^{+\infty}\lambda(k) = +\infty, \lambda(k+1) \leq \lambda(k), \lim_{k\to+\infty}\lambda(k) = 0$.

By using the relaxation sequence stated above, the POCS will converge to a point in G, which has been verified by simulation in [5].

As addressed above, the POCS method can be used for source localization in both consistent and inconsistent cases as long as an appropriate relaxation sequence is applied. However, for sequential distributed algorithms, a specified data transmitting path is demanded, i.e., a cyclic form. How to do path planing in such networks is also a big issue. Also, the convergence rate is low when the sensor density is large and data transmission becomes unreliable when some of the nodes fail. Hence the robustness of such networks is low.

B. Parallel Projection Method

The parallel projection method is a centralized algorithm, where each sensor transmits its measurement to a fusion center. The parallel projection method involves finding a point θ^* which minimizes the weighted sum of the squares of the distances to the convex set $\mathcal{D}_i$s.

$$\theta^* = \arg\min_{\theta\in\mathbb{R}^2} \sum_{i=1}^{N} w_i \|\theta - \mathcal{P}_{\mathcal{D}_i}(\theta)\|^2, \tag{8}$$

where $(w_i)_{1\leq i\leq N}$ are strictly positive weights such that $\sum_{i=1}^{N} w_i = 1$.

We assume G' is the set of weighted least-square solutions of the convex feasibility problem in (8). The algorithm for the parallel projection method is given as follows

$$\theta(k+1) = \theta(k) + \lambda(k)\left(\sum_{i=1}^{N} w_i \mathcal{P}_{\mathcal{D}_i}(\theta(k)) - \theta(k)\right), \tag{9}$$

where $\lambda(k)$ is a relaxation sequence.

Theorem 2: [8] Suppose sequence $\lambda(k)$ is in $[\epsilon, 2-\epsilon]$, where $0 < \epsilon < 1$. Then any sequence $\theta(k)_{k\geq 0}$ generated by (12) converges to a point in G'. □

The advantage of the parallel projection method is that it has a good convergence performance especially for the inconsistent cases of CFP. Also we can design a weight sequence $(w_i)_{1\leq i\leq N}$ according to the reliability of each sensor's measurements, i.e, we can assign a higher weight to the sensors with large received powers.

IV. Diffusion Based Projection Method

To avoid the path planning problem in the sequential distributed method and improve the robustness of the networks, a diffusion based approach is useful. In diffusion based methods, no fusion center is required and the sensor nodes need to communicate only with their closest neighbors. So it reduces the probability of congestion around the sink nodes and increases the robustness of the network against node failures or unpredictable switches to sleeping mode where data transmit path planning is not demanded.

Before we address the diffusion method, first we give a brief introduction about the diffusion network. Let us represent the diffusion network as an undirected graph defined by $\mathcal{G} := (\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is node set $\mathcal{N} := 1, \ldots, N$ and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the edge set. If node k can directly send data to node l, we define the undirected link by $(k, l) \in \mathcal{E}$.

If the source localization problem fall in the consistent case, in the literature, there already has a diffusion based method can be used which is proposed by A. Nedić and A. Ozdaglarin [9]. It works as follows: sensor i at time $k+1$ generates its estimate updates according to the following protocol

$$\theta^i(k+1) = \mathcal{P}_{\mathcal{D}_i}\left(\sum_{j=1}^{N} w^i_j(k)\theta^j(k)\right). \tag{10}$$

where $w^i_j(k), i = 1, \ldots, N, j = 1, \ldots, N$ denotes the weight; $\theta^i(0), i = 1, \ldots, N$ is arbitrary.

Assumption 1: (Network connectivity) The network is connected, *i.e.*, there exists a direct or indirect path between any two nodes in the networks.

Assumption 2: (Weighting rule) $W(k)$ whose i-th row is the vector $w_i(k) = [w^i_1(k), \ldots, w^i_N(k)]$, is an $N\times N$ doubly stochastic weighting matrix with the following properties:

$$\mathbf{1}^T W(k) = \mathbf{1}^T, W(k)\mathbf{1} = \mathbf{1}, \tag{11}$$

Theorem 3: [9](Consensus) Let the intersection set $\mathcal{D} = \bigcap_{i=1}^{N} \mathcal{D}_i$ be nonempty and Assumptions 1 and 2 hold. Then for some $\theta^* \in \mathcal{D}$, $\lim_{k\to\infty}\|\theta^i(k) - \theta^*\| = 0, \forall i = 1, \ldots, N$. □

From Theorem 3, we can see that in the consistent case, the protocol (10) has a good convergence performance. We can see that the estimates of every sensor will converge to the same optimal solution θ^* which is in the intersection of convex sets determined by each sensor. However, its convergence behavior in the inconsistent case is generally unsatisfactory and a general protocol for both consistent and inconsistent cases is needed.

A. Protocol for the General Case

As stated above, we assume $\mathcal{D} \neq \emptyset$. However, this assumption does not hold in the presence of observation noise which may lead to the intersection $\mathcal{D}$ being empty. To deal with the inconsistent case, we can force the assumption that $\mathcal{D} \neq \emptyset$. For example, we can expand $\mathcal{D}_i$ to increase the probability that the intersection of the convex sets determined by each sensor is not empty. However since we don't know how much expansion we should do, the estimation error will increase even though the algorithm does converge.

Instead, we propose a new protocol which works for both the consistent and inconsistent cases:

$$\theta^i(k+1) = \sum_{j=1}^{N} w^i_j(k)\theta^j(k) + \beta(k)\left[\mathcal{P}_{\mathcal{D}_i}\left(\sum_{j=1}^{N} w^i_j(k)\theta^j(k)\right) - \sum_{j=1}^{N} w^i_j(k)\theta^j(k)\right], \tag{12}$$

where $\beta(k)$ is the relaxation sequence, $w^i_j(k)$ is the same as addressed as in (10). We can see that if $\beta(k)_{k\geq 0} = 1$, then the protocol (12) is the same as (10).

B. Convergence Behavior of Protocol (12) for the Consistent Case

Before proving the convergence of the algorithm, we define some variables described as follows:

$$v^i(k) = \sum_{j=1}^{N} w^i_j(k)\theta^j(k). \tag{13}$$

Assumption 3: $\sum_{k=1}^{\infty} \beta(k)(1-\beta(k)) = \infty$.

Note that Assumptions 3 is justified. It can be easily checked that if $\beta(k) \in (0,1)$, then Assumption 3 holds.

From protocol (12), the relation between $\theta^i(k+1)$ and $\theta^1(s), \ldots, \theta^N(s)$ at time $0 \leq s \leq k$ is given by

$$\begin{aligned}\theta^i(k+1) = &\sum_{j=1}^{N} [W(k,0)]^i_j \theta^j(0) + \sum_{m=1}^{k} \beta(m-1)\\ &\left(\sum_{j=1}^{N} [W(k,m)]^i_j (\mathcal{P}_{\mathcal{D}_j}(v^j(m-1)) - v^j(m-1))\right)\\ &+ \beta(k)(\mathcal{P}_{\mathcal{D}_i}(v^i(k)) - v^i(k)).\end{aligned} \tag{14}$$

where $W(k,m) = W(m)W(m+1)\ldots W(k-1)W(k)$ is transition matrices. $W(k,m)^i_j$ denotes the (i,j)-th entry of $W(k,m)$.

Theorem 4: (Consensus) Let the intersection set $\mathcal{D} = \bigcap_{i=1}^{N} \mathcal{D}_i$ be nonempty and Assumptions 1-3 hold, then for some $\theta^* \in \mathcal{D}$, $\lim_{k\to\infty} \theta^i(k) = \theta^*, \forall i = 1, \ldots, N$.

The proof is omitted because of the limitation of the space.

From Theorem 4, we can see that the estimates of every sensor will converge to the same optimal solution θ^* which is in the intersection of convex sets determined by each sensor.

C. Convergence Behavior of Protocol (12) in Inconsistent Case

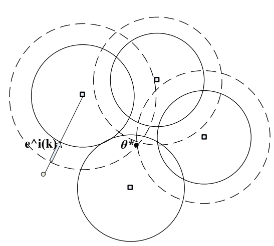


Fig. 3. Inconsistent case

In section IV-*B*, we provide convergence analysis of the proposed protocol (12) in the consistent case. In this subsection, we will derive conditions for convergence in the inconsistent case. First, we make the following assumption.

Assumption 4: $\sum_{k=1}^{\infty} \beta(k)(1-\beta(k)) = \infty, \beta(k+1) \leq \beta(k), \lim_{k\to+\infty} \beta(k) = 0$.

Let Assumptions 1, 2 and 4 hold, and suppose that the protocol (12) can converge to a point $\theta^* \in G$, where G is the set of least square solutions of (4). Then we define the new convex sets by $\mathcal{D}'_i, i = 1, \ldots, N$, dashed circles as shown in Fig. 3. Further we define the sequence $\{e^i(k)\}$ as the projection error as follows:

$$e^i(k) = \mathcal{P}_{\mathcal{D}_i}[v^i(k)] - \mathcal{P}_{\mathcal{D}'_i}[v^i(k)] \tag{15}$$

Then the protocol (12) can be cast as

$$\theta^i(k+1) = v^i(k) + \beta(k)\left[\left(\mathcal{P}_{\mathcal{D}'_i}[v^i(k)] + e^i(k)\right) - v^i(k)\right]. \tag{16}$$

Theorem 5: Suppose that $\sum_{k\geq 0}\sum_{i=0}^{N} \beta(k)\|e^i(k)\| < +\infty$ and Assumptions 1, 2 and 4 hold, then $\lim_{k\to\infty} \mathcal{P}_{\mathcal{D}'_i}(v^i(k)) = v^i(k), \forall i$.

Proof:

$$\begin{aligned}\theta^i(k+1) - \theta^* = &(1-\beta(k))(v^i(k) - \theta^*)\\ &+ \beta(k)(\mathcal{P}_{\mathcal{D}'_i}(v^i(k)) - \theta^*) + \beta(k)e^i(k)\end{aligned} \tag{17}$$

$$\begin{aligned}\sum_{i=1}^{N} \|\theta^i(k+1) - \theta^*\| \leq &(1-\beta(k))\sum_{i=1}^{N} \|v^i(k) - \theta^*\|\\ &+ \beta(k)\sum_{i=1}^{N} \|\mathcal{P}_{\mathcal{D}'_i}(v^i(k)) - \theta^*\| + \beta(k)\sum_{i=1}^{N} \|e^i(k)\|\\ \leq &\sum_{i=1}^{N} \|\theta^i(k) - \theta^*\| + \beta(k)\sum_{i=1}^{N} \|e^i(k)\|\end{aligned} \tag{18}$$

Since $\sum_{k\geq 0}\beta(k)\sum_{i=1}^{N} \|e^i(k)\| < \infty$, the series $\sum_{i=1}^{N} \|\theta^i(k+1) - \theta^*\|$ converges.

Next we define $\eta^i(k) = (1-\beta(k))v^i(k) + \beta(k)\mathcal{P}_{\mathcal{D}'_i}(v^i(k))$, then we have

$$\begin{aligned}\|\theta^i(k+1) - \theta^*\|^2 = &\\ &\|\eta^i(k) - \theta^*\|^2 + 2\beta(k)\langle \eta^i(k) | e^i(k)\rangle + \beta(k)^2\|e^i(k)\|^2\\ \leq &\|\eta^i(k) - \theta^*\|^2 + 2\left(\|\eta^i(k) - \theta^*\| + \beta(k)\|e^i(k)\|\right)\|e^i(k)\|\\ \leq &\|\theta^i(k) - \theta^*\|^2 - \beta(k)(1-\beta(k))\|\mathcal{P}_{\mathcal{D}'_i}(v^i(k)) - v^i(k)\|^2\\ &+ 2\left(\|\eta^i(k) - \theta^*\| + \beta(k)\|e^i(k)\|\right)\beta(k)\|e^i(k)\|\end{aligned} \tag{19}$$

Hence,

$$\begin{aligned}\sum_{i=1}^{N} \|\theta^i(k+1) - \theta^*\|^2 \leq &\sum_{i=1}^{N} \|\theta^i(k) - \theta^*\|^2\\ &- \beta(k)(1-\beta(k))\sum_{i=1}^{N} \|\mathcal{P}_{\mathcal{D}'_i}(v^i(k)) - v^i(k)\|^2\\ &+ B\sum_{i=1}^{N} \beta(k)\|e^i(k)\|\end{aligned} \tag{20}$$

where $B = \sup_{k\geq 0}\left(\|\eta^i(k) - \theta^*\| + \beta(k)\|e^i(k)\|\right) < +\infty$.

From (20), as $\sum_{k\geq 0}\beta(k)(1-\beta(k)) = +\infty$, we can get

$$\sum_{k\geq 0} \beta(k)(1-\beta(k))\sum_{i=1}^{N} \|\mathcal{P}_{\mathcal{D}'_i}(v^i(k)) - v^i(k)\|^2$$
$$\leq \sum_{i=1}^{N}\|\theta^i(0)-\theta^*\|^2 + B\sum_{k\geq 0}\left(\beta(k)\sum_{i=1}^{N}\|e^i(k)\|\right) < +\infty$$
$$\Rightarrow \lim_{k\to\infty}\sum_{i=1}^{N}\|\mathcal{P}_{\mathcal{D}'_i}(v^i(k)) - v^i(k)\|^2 = 0$$
$$\Rightarrow \lim_{k\to\infty}\mathcal{P}_{\mathcal{D}'_i}(v^i(k)) = v^i(k) \quad (21)$$

■

Similar to (14),

$$\theta^i(k+1) = \sum_{j=1}^{N}[W(k,0)]^i_j\theta^j(0) + \sum_{m=1}^{k}\beta(m-1)$$
$$\left(\sum_{j=1}^{N}[W(k,m)]^i_j(\mathcal{P}_{\mathcal{D}'_j}(v^j(m-1)) + e^j(m-1) - v^j(m-1))\right)$$
$$+\beta(k)\left(\mathcal{P}_{\mathcal{D}_i}(v^i(k)) + e^i(k) - v^i(k)\right). \quad (22)$$

Theorem 6: Let Assumptions 1, 2 and 4 hold, then

- $\lim_{k\to\infty}\theta^i(k) = \theta^j(k), i, j \in [1:N]$.
- if $\lim_{k\to k'}\mathcal{P}_{\mathcal{D}_i}v^i(k) = v^i(k), \forall i$, then $\lim_{k\to\infty}\theta^i(k) = \theta^* + Er$, where Er is proportional to $\beta(k'), \forall i$

Interpretation on the Condition $\sum_{k\geq 0}\sum_{i=0}^{N}\beta(k)\|e^i(k)\| < +\infty$.

From Theorem 5 and Theorem 6, we provide a theoretical basis that if $\sum_{k\geq 0}\sum_{i=0}^{N}\beta(k)\|e^i(k)\| < +\infty$ is satisfied, then the estimate of the sensors will converge to some point $\theta^* \in G$. In other words, $\theta^* \in G$ is some point which makes the assumption $\sum_{k\geq 0}\sum_{i=0}^{N}\beta(k)\|e^i(k)\| < +\infty$ hold.

Now, we give some explanations on the projection error $e^i(k)$. When the weight estimate (each sensor and its neighbors) $v^i(k)$ falls in both the convex sets $\mathcal{D}_i$ and $\mathcal{D}'_i$, $\mathcal{P}_{\mathcal{D}_i}[v^i(k)] = \mathcal{P}_{\mathcal{D}'_i}[v^i(k)]$ which implies $\|e^i(k)\| = 0$. Otherwise, $\|e^i(k)\|$ is the absolute difference between two radius, i.e., $\|e^i(k)\| = \|r_{D_i} - r_{D'_i}\|$, where r_{D_i}, and $r_{D'_i}$ denote the radius of the convex sets $\mathcal{D}_i$ and $\mathcal{D}'_i$ respectively. Intuitively, this difference is related to the measurement noise. However, it is difficult to justify the value of $\sum_{i=0}^{N}\beta(k)\|e^i(k)\|$. In our simulations, we have observed that $\|e^i(k)\|$ is either equal to 0 or a small value.

From the analysis of the property of $\sum_{i=0}^{N}\beta(k)\|e^i(k)\|$, we can see that the condition $\sum_{k\geq 0}\sum_{i=0}^{N}\beta(k)\|e^i(k)\| < +\infty$ is a little strong. However, the potential instability for this convergence condition can be avoided by using a sufficiently small relaxation sequence $\beta(k)$. For example, if the variance of receiver noise is small, we can just set the relaxation sequence as $\beta(k) = 1/k$. Otherwise, we may set it as $\beta(k) = a\times(1/k)$, where a is a small constant value which is related to the variance of receiver noise. Alternately, we can adopt the strategy for choosing $\beta(k)$ as done in [5]. At the first phase, the relaxation sequences are set to 1. Then if convergence to a limit cycle is detected, i.e., each sensor converges to a different value, the method enters phase 2. At phase 2, the relaxation parameters are decreased at a rate of $1/k$. Please note that after the method enters phase 2, the probability of $e^i(k) = 0$ is also increased. This will lead to the condition stated above hold for convergence.

Remark 1: From the analysis for both consistent and inconsistent cases, we can see that the proposed diffusion based algorithm can be applied to the source localization problem without prior knowledge whether or not the problem is consistent by using a deceasing relaxation sequence $\beta(k)$ as defined in Assumption 4.

D. Estimation Accuracy Analysis

Noiseless case: It can be easily seen that if the sensor observation is noiseless, then the algorithm can converge to the true source location as long as there are at least three non-collinear sensors in the sensing field.

Noisy case: For the noisy cases, our estimator $\min_{\theta\in S}\sum_{i=1}^{N}\|\theta - \mathcal{P}_{\mathcal{D}_i}(\theta)\|$ is optimal in the least square (LS) sense. Obviously, if the intersection of convex sets determined by each sensor is nonempty, i.e., $\mathcal{D} \neq \emptyset$, then $\min_{\theta\in S}\sum_{i=1}^{N}\|\theta - \mathcal{P}_{\mathcal{D}_i}(\theta)\| = 0$. Also according to the fact that "The intersection of closed sets is closed", we can see that because the convex sets (disks) determined by sensors are closed sets, the intersection of these sets is closed. Thus when the number of sensors increases to infinity, the variance of the estimation will decrease to zero, i.e., $\mathbf{E}(\theta^* - \bar{\theta}^*)^2 = 0$, where $\bar{\theta}^*$ denotes the mean value of θ^*. If the intersection of convex sets determined by each sensor is empty, i.e., $\mathcal{D} = \emptyset$, besides the number of sensors, there are two other factors affecting the estimation accuracy. One is the geometrical configuration of the senor field and the other is the sensor observation noise.

Remark 2: Throughout this paper, we have assumed that channel between sensors is perfect without transmission noise and a node can transmit perfectly and reliably without packet loss to its neighbors. However, in many situations this is not realistic and will be considered as our future work.

V. Simulations

This section presents simulation experiments for a sensor network with 15 sensor nodes randomly placed in a 10 m × 10 m field. At each sensor, a measurement of the source energy is generated according to (1). The gain factors $g_i, i = 1,\ldots,N$, for all sensors are equal to 1. The source is located at $\theta = [0,0]^T$ and emits a signal with P set to 50 dB and the background noise level is set at $0 \leq \sigma_i \leq 2, i = 1,\ldots,N$ for all sensors in the sensor field. The actual receiver SNR at different sensors depends on the sensor to source distance. For example, if the variance of noise is 1, then for a sensor that is 5 m away from the source, its receiver SNR is $10\times\log_{10}(50/5^2) = 3$ dB.

A. Convergence Performance of the Proposed Methods

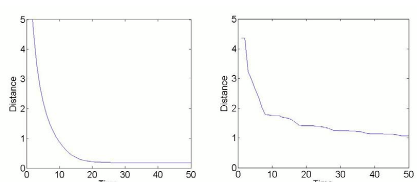
Due to the measurement noise, in our simulation, the CFP is of inconsistent case. The relaxation sequences for parallel and sequential projection methods are set as $\lambda(k)_{k\geq 0} = 1$(parallel) and $\lambda(k)_{k\geq 0} = 1/(k+1)$ (sequential) respectively. Figures 4 shows the simulation results for these two methods, where the number of active sensors is 10. We can see that both the parallel and sequential projection methods have good convergence performance.

For the diffusion based projection method, the number of active sensors is also set to 10 and communication range of each sensor is set as 5m to guarantee network connectivity (Assumption 1). The variance of background noise is 1. The relaxation sequence is set as $\lambda(k)_{k\geq 0} = 1/(k+1)$. The weight matrix used is Metropolis weights presented in [6]. Fig. 5 (left) shows one example of network connectivity. In Fig. 5 (right), each disk denotes one convex set determined by a sensor, i.e., $(\mathcal{D}_i)_{(i=1,\ldots,N)}$. Clearly, $\mathcal{D} = \bigcap_{i=1}^{N} \mathcal{D}_i$ is empty. Fig. 6 presents the convergence result by using protocol in [9] and the proposed protocol, where each curve denotes the convergence result of one sensor and distance (y-coordinate) denotes Euclidean distance between the estimated and true source location, i.e., $\|\hat{\theta}^i - \theta\|, \forall i$. From the figure, we can see that our proposed diffusion based projection method has good convergence performance even though the CFP is inconsistent. However, in most of the cases, the protocol (10) in [9] will diverge or oscillate at some point.

B. Estimation Performance of the Proposed Methods

As a benchmark, the performance of the proposed projection methods against weighted least square (WLS) method in [4] and MLE in [1], is also conducted. The MLE is found by performing a grid search over the field area. In our implementation, the grid search resolution is set to 0.1 m × 0.1 m. The performance of the estimators is evaluated through 1000 trials.

As shown in Fig. 7, our proposed projection based methods have a comparable estimation accuracy with MLE when the noise level is low or the number of active sensors is large. Also, we can see the performance of projection based methods are better than WLS, which is mainly because least square based methods are sensitive to the noise and need a high density of sensors.



(a) Parallel (b) Sequential

Fig. 4. Convergence results by Parallel and Sequential projection methods.

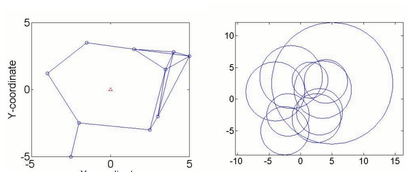
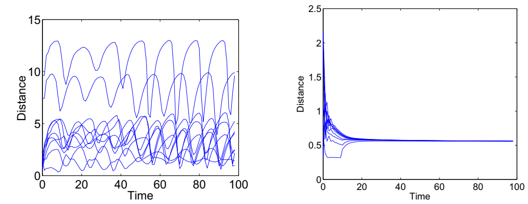
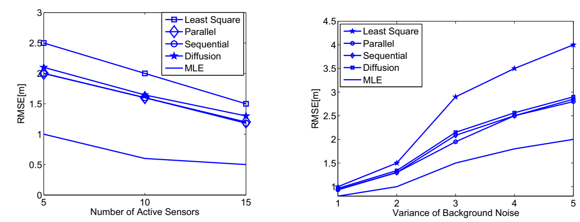


Fig. 5. Sensor network connection (left) and convex sets (disks) determined by sensors (right).



(a) Protocol of [9] (b) Proposed protocol (12)

Fig. 6. Convergence results by diffusion protocols



(a) Number of the sensors v.s. RMSE (b) Noise v.s. RMSE

Fig. 7. Accuracy comparison

VI. Conclusions

In this paper, we proposed a diffusion based projection method for energy based source localization problems. The solution has global convergence properties and acceptable estimation accuracy. The convergence of the proposed protocol was provided. Theoretic analysis and simulation results showed that our proposed method can be applied to the distributed source localization problem without prior knowledge of whether or not the problem is consistent. Future works include studying the diffusion method for source localization with channel noises and packet loss.

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